Deepening Our Roots: A Philosophical Approach to Teaching and Learning Polynomials

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West Charlotte High School

This curriculum unit is recommended for:

North Carolina Math 3

Keywords: Polynomials, Philosophy, Fundamental Theorem of Algebra, Remainder Theorem, Discovery-Based Instruction, Problem-Based Learning

Teaching Standards: See Appendix 1 for teaching standards addressed in this unit.

Synopsis: Polynomial expressions are one of the largest and most important topics of North Carolina Math 3; however, in addition to being important for student outcomes in Math 3, they are foundational to higher math classes such as pre-calculus and AP Calculus. My unit takes advantage of polynomial functions as the ideal place to increase rigor of instruction and student discussion. By merit of being relatively well-behaved functions, factoring and solving polynomials, the fundamental theorem of algebra, and remainder theorem all loan themselves well to instruction through discovery and discussion. Using philosophy as a lens, my curriculum unit looks to increase student agency in noticing and naming structures with polynomial expressions and synthesizing these relationships to approach problems with multiple approaches as opposed to viewing the standards of the unit as unrelated problems.

I plan to teach this unit during the coming year to 60 students in NC Math 3

Introduction

Rationale

North Carolina Math 3 is the third and penultimate course in the North Carolina course of study for high school math. While Math 3 covers a diverse range of topics, including functions, polynomials, rational functions, exponential equations, trigonometry, geometry, and statistics, the course does contain several *power standards*, which are more heavily tested on the End-of-Course (EOC) final assessment; additionally, these topics provide an important foundation for the skills and knowledge necessary to succeed in upper-level math, such as International Baccalaureate (IB) Mathematics, Pre-calculus, Math 4, and Advanced Placement (AP) Calculus coursework. Polynomial functions are one such set of power standards; mastery of Math 3's polynomial content is immediately important for student success on their EOC assessment and also provides critical conceptual and procedural knowledge for advanced coursework in the future. Teaching polynomials in a manner that is aligned to the standards set by the NCDPI and the rigor of the EOC exam is critical to student success; during my teaching career at West Charlotte High School, I have generally had success with teaching a well-aligned course whose rigor reflects that of the Math 3 EOC final exam.

One area of weakness with my instruction; however, has been instructional rigor--specifically on polynomial standards, which often can be highly procedural. For this reason, I have come to rely on direct instruction for a large portion of this unit, focused on methods that favor efficiency and fluency. While teaching procedurally-heavy material through direct instruction and solidifying student mastery with practice exercises can quickly help students demonstrate proficiency on most topics in Math 3, this can create conceptual gaps that leave students viewing the topics of Math 3's polynomials unit as a series of unrelated problems as opposed to seeing a multitude of connections between various skills. The scope and sequence of polynomial topics in Math 3 provides opportunities for students to connect crucial mathematical concepts, demonstrate creativity by taking multiple approaches to solving a relatively straightforward problem, become competent sense-makers by seeing and justifying structures without direct instruction, and demonstrate a higher level of mastery than the EOC exam by justifying and explaining solutions.

My curriculum units' goals are twofold, both of which will utilize elements of philosophy to achieve these ends. First, students will be expected to achieve a higher level of mastery by setting the goal of justifying, explaining, and extending solutions (Webb's Depth-of-Knowledge Level 4), as opposed to the questions involving basic recall (DOK 1), procedural competence (DOK 2), or strategic thinking (DOK 3) that are most prevalent on the Math 3 EOC exam. My curriculum unit will utilize content research on formal logic and framework for explanation and justification to achieve this objective. Second, students will be held to a higher level of instructional rigor by engaging in lessons that will utilize discovery-based learning that postures the teacher as a facilitator for their seeing structure, as opposed to acting as the sole gatekeeper of information. This objective will be informed by philosophical content research on models of pedagogy, education as the synthesis of experience, and best practices for diverging from primarily using direct instruction when possible. In doing so, I hope to provide teachers with intellectual framework and concrete activities to give students tools to demonstrate mastery on the standards as they appear on the Math 3 EOC Exam,

Demographics

West Charlotte High School is a public, Title 1 high school in Charlotte Mecklenburg Schools' Central 1 learning community. Founded in 1938, West Charlotte High School was the only all-black school in Charlotte Mecklenburg Schools to remain open following the Swann v. Charlotte-Mecklenburg Board of Education court case that desegregated CMS in 1971. Following the Swann verdict, West Charlotte became the first formerly all-black school to be integrated with white students in the country; during this time period, West Charlotte High School was a point of pride in Charlotte, at that point known as *the city that made integration work*. Following the Leandro v. State of North Carolina verdict in 1997, Charlotte-Mecklenburg Schools began to re-segregate, which included West Charlotte High School.

As such, West Charlotte High School's student body is predominantly comprised of students of color: 80% of students identify as African-American, 12.9% Hispanic, 3.5% Asian, 2.1% biracial, and 1.5% White. Additionally, West Charlotte has a significantly higher proportion of economically disadvantaged students relative to the district and state: 57.7% of students at West Charlotte High School compared to 43.4% of students in the state. Other proxy statistics to child poverty point to the prevalence of socioeconomically disadvantaged: 99.7% of West Charlotte students qualify for free lunch and 4.4% of students are McKinney-Vento. As a result of this, West Charlotte's performance looks lackluster at the surface-level: a chronic absenteeism rate four times higher than the district average, suspension rate over five times the district average, and graduation rate (75.7%) that is significantly below the district average (85.5%). While these statistics indicate that West Charlotte has significant issues, they don't tell the entire story of the school as it will shape my approach to writing and teaching this curriculum unit.

West Charlotte High School is currently partnered with Equal Opportunity Schools (EOS) with the goal of increasing student enrollment in advanced coursework such as AP classes in order to bridge the opportunity gap between traditionally underrepresented students and rigorous coursework. Additionally, West Charlotte High School has received the designation of a magnet school due to its full IB program, which received its Middle Years Program (MYP) certification during the 2020-21 school year. As was outlined in my rationale, I will be writing a unit with the goal of high instructional rigor, as well as rigor in assessment; the main influence my school demographics will have will be my unit taking special care to scaffold key prerequisite knowledge for students that are not yet grade level proficient.

Objectives

My unit will be aligned to the NC Math 3 standards that primarily cover polynomial functions. The standards are listed and annotated in Appendix 1 for teachers' reference. The topics include polynomial division (NC.M3.A-APR.6), the factor and remainder theorem (NC.M3.A-APR.2), factoring and solving higher order polynomial expressions (NC.M3.A-APR.3), writing the equations of higher-order polynomials (NC.M3.A-SSE.2), applying the fundamental theorem of algebra (NC.M3.N-CN.9), and describing the key features of polynomial functions (NC.M3.F-IF.7). These standards can also require review of content from Math 1 and Math 2 as prerequisite knowledge and provides opportunities for extension activities that delves into Pre-Calculus content.

Content

The scope of my research has been threefold. First, I have done mathematical research into the standards of my curriculum unit to provide teachers with a deeper understanding of the content which they will be teaching. In order for students to achieve the Webb's DOK 4 level of mastery outlined in my rationale of explanation, hypothesis, and extension, their teachers must possess sufficient content knowledge in order to model the same. This research includes math history, proofs, and instructional analogies that will allow teachers to present polynomial functions as well-behaved relationships that are consistent with many of the prior skills and concepts that students have seen before. Second, my unit will summarize philosophical content on the nature of education, learning, and discourse. Part of this will include writings on education as the synthesis of experience, formal logic, and construction of arguments that align to my goals of DOK level 4 rigor as the measure of student success. Finally, I have researched both mathematical and general pedagogy that connects to the content and philosophical methods outlined. This serves to meet my goal of increasing the instructional rigor of the lessons which will deliver the content to students. With these components, I hope to provide teachers with an easy-to-implement curriculum unit that makes content as coherent as possible; additionally, my lessons will the necessary instructional scaffolds for students with the goal of supporting even students who are below grade level to achieve a level of mastery that includes explanation, hypothesis, extension, and justification.

In order to plan rigorous instruction that implements elements of philosophy around the standards of polynomials in Math 3, it is first necessary to understand the nature of polynomial functions, how mathematicians' understanding of these expressions has evolved over time, and connect polynomial operations with prior knowledge that can serve as a useful analogy for instruction. It is the goal of my unit to augment the typical definition that alludes primarily to the algebraic representation of these equations, such as the one presented in textbooks such as *Bridges to Algebra and Geometry*: "A polynomial is a variable expression containing one or more terms. Polynomials are named by the number of terms. A monomial has one term; a binomial has two terms; a trinomial has three terms[1]." While the textbook does go on to present some of the concepts I will explore in my unit, they primarily surface-level and typically are abandoned by Math 3 in favor of representations that favor procedural efficiency as opposed to a deeper conceptual understanding.

Since middle school math, students, have learned operations with polynomial expressions, including addition, subtraction, multiplication, and division (with a single term, or monomial; division by an expression of multiple terms is covered in Math 3). While polynomial operations are not technically in the standards of Math 3, I typically spend one day covering these topics. This is to build fluency and correct deficits in prerequisite knowledge critical to three Math 3 topics: polynomial long division (NC.M3.A-APR.6), writing polynomial equations from zeros (NC.M3.A-SSE.2), and operations with rational functions (NC.M3.A-APR.7). Review of this procedural knowledge can feel dry, boring, and excessively abstract when limited to only

[1] Brightes to Algebra and Geometry 654.

algebraic representations of operations. For this, there are two solutions I propose to create more meaningful learning experiences around this review.

First, is to connect polynomial operations—addition, subtraction, multiplication, and (specifically long) division—with the procedures for those same numeric operations. This will not only enforce procedural fluency by connecting areas with potential deficits to prior knowledge from elementary school, but also provide opportunities for writing and discussion prompts comparing problems, which can be used to increase the rigor of lessons. This idea was partially by presented by Cornelis de Groot and Meredith Boyajian for addition of polynomials, specifically as it relates to the definition of like terms. De Groot and Boyajian critique the common practice of math teachers to lean heavily on pseudoanalogies for addition of polynomials, which requires students to combine *like terms*[2] (i.e. the terms of a polynomial must have the same exponent and variable; for instance, an x term could only be added to another x term and an x^2 term could only be added to another x^2 term). This can be detrimental to students seeing structure in expressions and, while analogies divorced of mathematical context such as "apples and oranges," can build procedural fluency in the short term, creates long-term barriers to meaningful conceptual understanding. One relatively simple solution to this issue is to simply connect polynomial addition and like terms to numeric addition to the context of addition in terms of ones, tens, hundreds, etc. places; doing so gives students both a sense of structure in math and contributes to procedural fluency by activating prior knowledge (see Appendix 3). This allows teachers to connect students' experience with placeholders and numeric operations to more seamlessly translate into algebraic structures; a student would not add ones and tens, and thus it would make sense not to combine a constant with an x term.

Polynomial multiplication is an additional operation that can be similarly connected to numeric operations; however, doing so in a manner that maintains procedural fluency for students and allows for more multiplication of more complicated problems may require the teacher to re-frame several procedures, depending on what students learned in elementary and middle school. The "FOIL" method of multiplication is oft-used to teach both multiplication of two-digit numbers and binomial expressions (i.e. polynomials of two terms); in doing so, students will multiply the first, outer, inner, and last terms of an expression, then add the terms together. For instance, the numeric multiplication 58×14 could be calculated as $(50 \times 10) + (50 \times 4) + (8 \times 10) + (8 \times 4) = 812$ using the FOIL methodology; similarly, the binomial expression $(x-1)(x+5)$ would be written as $(x \cdot x) + (x \cdot 5) + (-1 \cdot x) + (-1 \cdot 5) = x^2 + 5x - x - 5 = x^2 + 4x - 5$. While this strategy avoids the critiques that de Groot and Boyajian make towards teaching strategies of like terms through activating prior experience by highlighting the common structures and procedures between numeric and polynomial expressions, this particular method is unwieldy as students start to multiply larger expressions of more terms; for instance $(x+3)(x^2+10x-2)$. Isaac Frank summarizes this issue and proposes an alternative that most high school math teachers already use called the "box method" of multiplication[3]. Procedurally, the box method builds procedural fluency without sacrificing connections to the distributive property of multiplication; however, Frank notes that using this method relates to geometric representations of multiplication that show numeric products as the area of a rectangle to

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[2] Cornelis de Groot and Meredith Boyajian. 2015. "Like Terms: What's in a Name?" *The Mathematics Teacher,* Vol 108 (7): pp. 506-511.

[3] Frank, Isaac, 2019. "When FOIL Falls Apart" *The Mathematics Teacher*, Vol 112 (6): pp. 460-1.

highlight the distributive property (see Appendix 3). Furthermore, I also will propose teaching numeric multiplication with de Groot and Boyajian's strategy of representing numeric expressions in base ten parallel to polynomial expressions in combination with the box method for polynomial multiplication (see Appendix 3). This adds another layer of experience that students can connect to polynomials to comprehend their structure and see them as an extension of their prior experience, rather than an unrelated problem.

Reviewing operations of polynomials in this manner will connect with John Dewey's metaphysical writings on education, that is to say, education being innately tied to the experience of the learner and the teacher's ability to levy these experiences is what creates learning and growth. In *Experience and Education*, John Dewey writes "The basic characteristic of habit is that every experience enacted and undergone modifies the one who acts and undergoes, while this modification affects, whether we wish it or not, the quality of subsequent experiences. For it is a somewhat different person who enters into them.... It covers the formation of attitudes, attitudes that are emotional and intellectual; it covers our basic sensitivities and ways of meeting and responding to all the conditions which we meet in living[4]." While Dewey takes on a more holistic approach to the possibilities and potential virtues of educators using experience to build habits, this has practical implications for the math educator to consider with respect to the goals of reviewing polynomial operations as well as the subsequent standards of Math 3. First, according to Dewey, it is not enough to simply leverage student experience—what makes an experience educational is that it helps students grow and sets the conditions for further growth[5]. The question "will an experience help a student grow?" and "will the experience create conditions for further growth?" are specifically answered when considering the varying routes that teachers take to build procedural fluency with polynomial addition and multiplication. As de Groot and Boyajian point out, teachers activating certain experiences through non-mathematical pseudoanalogies are, in fact, detrimental to seeing structure in expressions. Additionally, while methods such as FOIL for polynomial multiplication do have procedural and conceptual consistency but are less ideal for future growth than other avenues of activating prior experience that will hold up better to more complicated problems.

One aspect of polynomials that is more deeply explored in Math 3 is the relationship between polynomial equations and their zeros (also referred to as solutions, roots, and/or x-intercepts). A zero of a polynomial is any value that, when evaluated, will give the expression a value of zero. Typically, the most efficient way to solve for the zeros of a polynomial is to re-write the equation in terms of its factors, then solve for each of the factors of the expression. Due to the zero property of multiplication, it is often the case that zeros and factors of a polynomial are the opposite of each other (e.g. if $(x+1)$ is a factor of a polynomial expression, then it must have a zero at $x=-1$; conversely, a polynomial with a zero of $x=3$ must have $(x-3)$ as a factor). In the North Carolina high school math curriculum, students should be familiar with the process of factoring and solving quadratic polynomials from Math 1 and Math 2 already. While I typically review this topic for procedural fluency for the rational functions unit and as an opportunity to

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[4] Dewey, John. *Experience and Education*, 2

[5] Dewey 3-4.

re-emphasize the relationship between factoring and zeros before utilizing the concept with more complicated relationships in the Fundamental Theorem of Algebra and Remainder Theorem, I believe that there are opportunities here to present students with answers to some of the metaphysical questions of mathematics by exposing them to math history; specifically, ancient approaches to the same problems that they will be using to review prior content.

Prior to the advent of modern algebraic notation, Greek and Arab mathematicians utilized geometric conceptions of number and quantity. In Book 2 of *Elements*, Greek mathematician Euclid explores the shape of rectangles. In the definitions outlined in Book 2, Euclid defines a rectangle as the intersection of two pairs of parallel lines occurring at a right angle[6] (note that parallel lines are defined in the controversial Parallel Postulate of Book 1[7] and the construction of a perpendicular line is proved in proposition 11 of Book 1[8]). With these rigorous definitions, it is then that Euclid begins to lay the foundations for modern mathematics—specifically quadratic polynomials—by proving the properties of rectangles; this influence was so profound that quadratic equations are named after the quadrilaterals which Euclid studied and are an abstraction of a rectangle's area produced by the intersection of two lines (linear factors, as they are called in modern terminology). Euclid's proofs of rectangles would lay the groundwork for modern mathematical techniques to work with quadratic equations, including factoring by the greatest common factor (Book 7, Proposition 10), factoring quadratic expressions (Book 2, Proposition 7), factoring the difference of perfect squares (Book 2, Proposition 5), and completing the square (Book 2, Proposition 6). While reading the original text of Euclid's Elements would require knowledge of mathematical proofs and formal logic that is not contained in the scope or sequence of North Carolina high school mathematics through Math 3, a teacher studying, applying, and using rectangles as a concrete representation of algebraic expressions can make less abstract the dominant representations of certain polynomials in Math 3 coursework and provide some answers to the metaphysical questions that students may raise on course content (e.g. "What even is a polynomial?" What does this equation mean?").

While most math historians generally credit Hindu mathematicians in the fifth, sixth, and seventh century with the rudiments of algebra, it was Arab mathematicians' work that was especially notable for the development of the field, as well as its spread by way of the Arab Empires[9]. Most notable of the Arab mathematicians was Abu Ja'far Muhammed ibn Musa Al-Khwarizmi, who is typically credited with being most influential on the development of solving equations; his original name for the practice, *Kitab fi al-jabr w'al-mugabala*, is where the word *algebra* itself is derived, with *al-jabr* roughly translating to mean "balance" (on both sides of an equation) and *mugabala* referencing cancelling equal quantities from both sides of an equation[10]. Al-Khwarizmi's work covered solving methods for linear and quadratic equations, although his representation and verbiage is dramatically different than is commonly shown to high school students. First, Al-Khwarizmi did not use symbols, rather all of his equations and solving

[6] Euclid, *Elements*, 50.
[7] Euclid 28
[8] Euclid 16
[9] Krantz 55.
[10] Krantz 56.

techniques were done with words; a square being analogous to x^2, a root being analogous to x, and numbers simply being known quantities[11]. With this verbiage, Al-Khwarizmi solves quadratic equations using the geometric properties of rectangles proved by Euclid. As such, even a relatively straightforward problem with modern methodology and notation such as $x^2 + 10x = 39$ can provide ample opportunity for answering some of the metaphysical questions of math that are raised when number and quantity are abstracted to the degree that is typical in modern high school mathematics[12] (see Appendix 5). While technically falling into the standards of Math 2, an adjacent extension of solving quadratics can further involve connecting Euclid's sixth proposition in Book 2 with the Quadratic Formula; a derived equation used in modern math to simplify solving quadratic expressions[13]. Proof of the quadratic formula is relatively straightforward and can be done easily using a direct proof; additionally, it reviews prerequisite knowledge of completing the square, which is necessary for future standards in Math 3, specifically converting circle equations into standard form (NC.M3.G-GPE.1) and solving polynomial equations (NC.M3.A-APR.3). In addition to foreshadowing future material, discussion of geometric representations of algebra and using them to prove the quadratic formula can be used to build student confidence for seeing, proving, and justifying patterns and structure throughout Unit 3 as was outlined as a goal of the unit.

When reviewing heavily procedural prerequisite knowledge in my polynomials unit, I have fallen into the trap of prioritizing fluency over conceptual coherence, as well as instructional rigor. I believe that integrating the outlined methods of connecting polynomial operational procedures to numeric operational procedures, as well as integrating historical approaches (including geometric representations) to problem-solving, that students can not only build procedural fluency on prerequisite knowledge, but also begin to more thoroughly understand the deeper conceptual connections of polynomials that are critical to demonstrating mastery on the standards of Math 3. A more thorough look at Euclid and Al-Khwarizmi's conceptions of rectangles, squares, and quadratic equations can ground student understanding on linear factors—a key concept for factoring polynomials, polynomial division, the remainder theorem, and the fundamental theorem of algebra—and build comprehensive understanding that avoids the unit from feeling like a series of unrelated problems. Additionally, explaining and justifying relationships—such as Young's graphic organizer (see Appendix 7)[14]--provides higher rigor of assessment and opportunity for students to achieve a higher level of mastery in explanation than doing.

Instructional Implementation

Teaching Strategies

My curriculum unit seeks to use instructional methods that are consistent with building a Community of Philosophical Inquiry (CPI), as defined by Lone and Burroughs. Establishing a

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[11] Krantz 58.
[12] Krantz 60
[13] Krantz 81
[14] Young, Donna, 2012. "A Graphic Organizer for Polynomial Functions," *The Mathematics Teacher,* Vol 106 (2), pp. 160-2.

CPI requires a classroom space that is conscious of the power dynamic between teacher and students, establishes epistemological modesty, respects student contributions, entertains a wide range of viewpoints, and models intellectual playfulness[15].

Direct instruction is a useful tool for math educators, especially for teaching procedural mathematics; however, my unit integrates alternatives that address some of the issues with the common "I do, we do, you do" format of teaching and caters to integrating elements of a CPI in the classroom. First, direct instruction with "I do, we do, you do," establishes the teacher as the sole authority on subject matter; a CPI shifts the teacher's role to that of a facilitator for inquiry as opposed to a gatekeeper of knowledge. Influential on my activities for the unit was Chrissy Allison's argument for integrating problem-based learning into the math classroom[16]. Allison and McDougal discuss some of the pitfalls of the "I do, we do, you do" method of instruction; namely, it fails to prepare students to utilize mathematical problem-solving on questions they haven't seen variations of before and that it does little to serve for deep, mathematical discussion that teachers typically state as their goal. Problem-based learning poses one potential solution to this issue; in problem-based learning, students are presented with a new problem they haven't seen before but hold all the prerequisite knowledge to solve. Students are then (crucially) given sufficient time to attempt the problem and justify their answers, before reconvening as a class to consider all methods used, including which ones draw on appropriate concepts and sense-making; steps for more efficiently solving similar exercises in the future are summarized as the synthesis of discussion.

Discovery activities are one such avenue for problem-based learning. While there is room for discussion whether discovery activities hijack student thinking too much by providing a curated series of prompts and questions to emphasize key relationships and concepts, I believe that there are certain benefits to utilizing a more structured activity. First, my discovery activities provide students with review of prerequisite knowledge; this allows teachers to integrate practice with new instruction, as well as address gaps in prior knowledge that may present themselves as a result of remote learning during the pandemic. Second, a more structured activity will make my curriculum more accessible to teachers, especially those without experience facilitating problem-based learning. Crucial to any discovery activity is the synthesis piece, which is summarized with direct instruction (teachers may also use this if they are uncomfortable with using more discovery activities). Some direct instruction may also be necessary to front-load the requisite vocabulary for students to comprehend questions, as well as have the appropriate language to streamline discussion of complicated mathematical concepts; while most of the language should be prerequisite knowledge, I have identified the breakdown of requisite vocabulary that teachers should be aware of checking their classes are fluent in the Classroom Lessons and Activities section.

Classroom Lessons and Activities

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15 Lone and Burroughs. *Philosophy in Education*, 60

16 Allison, Chrissy. "E48-Ditch I Do, We Do, You Do As Your Go-To-with Tom McDougal," *The Mindful Math Podcast*. May 2, 2021. https://www.audible.com/pd/E48-Ditch-I-Do-We-Do-You-Do-As-Your-Go-To-with-Tom-McDougal-Podcast/B093YSR6R9

My curriculum unit follows my traditional pacing for the polynomial standards in NC Math 3, consisting of 16 totals days, accounting for two flexible days for review, as well as a day for the unit 3 assessment (See Appendix 2 – Curriculum Unit Cycle Plan and Resources). As I have traditionally done in the past, my curriculum unit will start with five days reviewing prerequisite information outside the scope and sequence of NC Math 3 that is critical prerequisite knowledge the grade level standards and begins reviewing for the NC Math 3 rational expressions standards prior to instruction for those days. The prerequisite topics are operations addition, subtraction, and multiplication of polynomials (NC.M3.A-SSE.2), as well as factoring linear and quadratic polynomial expressions, including solving those equations (NC.M3.A-APR.3). Additionally, these five days should be used to build (or, more specifically, remind) students of the mathematical vocabulary that will be crucial for the rest of the unit.

Day 1 will introduce mathematical vocabulary specific to polynomial expressions to students with a twofold goal. First, pedagogically, defining mathematical jargon will hopefully streamline instruction for the teacher, and give both students and teacher the requisite terminology to describe some of the more abstract concepts in the unit. While in Math 1 and Math 2 teachers can typically lean on pseudoanalogies (e.g. "apples and oranges"), introducing concepts such as the Fundamental Theorem of Algebra and Remainder Theorem to students will be exceedingly difficult unless they have a clear, precise, and correct definitions of the terms root, zero, x-intercept, solution, (linear) factor, standard form, and factored form. While accessibility is certainly important, de Groot and Boyajian's analysis certainly highlights the problematic results of when teachers lean too heavily on layman's terms and dodge proper mathematical vocabulary[17]. Additionally, standardized tests on which students are being assessed for success (i.e. NC Math 3 EOC Final exam, SAT, and ACT) all utilize formal mathematical vocabulary and thus teaching out-of-alignment can create conditions where students aren't able to demonstrate mastery of skills and content due to a de facto language barrier, even if they hold some degree of proficiency. As was previously outlined in content, Dewey's framing of education producing growth and setting the conditions for future growth are both heavily considered in this unit and this is perhaps especially important on the first few days.

In addition to establishing mathematical vocabulary on the first day of the unit, my pacing also covers operations with functions, including function notation for the operations (i.e. $(f+g)(x)$ for addition, $(f-g)(x)$ for subtraction, $(f \cdot g)(x)$ for multiplication, and $(\frac{f}{g})(x)$ for monomial division; compositions of functions, $(f^{\circ}g)(x)$, should be taught with inverse functions as means of verifying inverses). Although students have seen and practiced operations with polynomials since middle school, in my experience this can be an area that lacks the fluency required for Math 3 standards that utilize these skills heavily (e.g. polynomial long division, writing polynomial equations from roots, operations with rational expressions). By reviewing operations early in tandem with vocabulary relevant to these skills (i.e. coefficient, term, variable, and exponent), teachers prime students to reference features of equations relevant to more complicated relationships. Additionally, function notation for operations with functions will benefit students preparing to take pre-calculus, as composite functions are more significant of a

[17] de Groot and Boyajian

topic than in Math 3. Finally, this serves as a place where students can begin to make some of the deeper connections to prior knowledge of numeric operations outlined earlier. By connecting polynomial addition and subtraction to numeric addition and subtraction to establish like terms, as well as showing the box method of multiplication for both numeric and polynomial products, teachers can begin to ground abstract expressions in concrete, numeric knowledge that students already possess (see Appendix 3: Connecting Polynomials with Numeric Operations). In light of learning loss due to covid-19, it is especially important for high school educators to reference material from elementary and middle school as much as possible, as the prerequisite knowledge that would traditionally be activated from Math 1 and Math 2 would be that which was most severely interrupted by the pandemic at the time of writing this unit.

Day 2 presents students with their first opportunity to begin guided, discovery-based learning with seeing patterns between key features of polynomial functions (i.e. their number of turns, end behavior, and x-intercepts) with the aspects of their equations that predict these features (i.e. degree for number of turns, lead coefficient and degree for end behavior, and multiplicity for the x-intercepts and their shape).While some of the vocabulary for discussing the equation should have been covered on the prior day, it will be important for teachers to abbreviate direct instruction to front-load some vocabulary to students through guided notes so that they have the language to more succinctly describe the relationship between the features of equations and their graphs. The discovery activity provides enough structure with the activities and questions so that students are given some direction towards significant aspects of the equation as they relate to key features (e.g. a question asking for the x-intercepts of the graph and whether they cross, bounce, or pass at a point of inflection is shown alongside the factored form of the equation); however, teachers looking to remove structure from the activity are free to remove or modify the scaffolds I have in place. Crucially, one of the most important aspects of discovery activities as the main vessel of instruction is discussion and summary of the results. I personally like to use guided notes so that students may easily name relationships and patterns, then reference those results in the future.

Day 3 is primarily intended for practice of the skills and content from days 1 and 2; however, as a task of higher-order thinking, I have included an exercise in formal logic and proofs for students. At the beginning (or end) of class, students are to be asked to prove that the product of two odd functions is odd and that the product of two even functions is even. Students to this point may have little to no experience with mathematical proofs; in Math 2, students are required to know and apply triangle congruence theorems, including writing a two-column proof; however, this hasn't been a power standard in the years of Math 2 being a tested subject and is not prerequisite knowledge to any of the power standards of Math 3. As a result, in my experience teachers at my school typically teach the theorems but spend little time on the formal reasoning required to arrive at these theorems (which, students usually wouldn't see anyhow until college math when studying Euclid's Elements), or the methodology for writing a two-column proof that makes use of them. Proofs are typically taught using geometry since geometric proofs require formal logic to arrive at theorems; however, they are concrete enough for students to easily grasp; however, even/odd proofs offer similar levels of concreteness and can be shown with direct proof (see Appendix 4: Even/Odd Function Proofs). In order to lead this activity successfully, teachers should have the final proof in-hand (including the lemma for sums of even and odd numbers), as well as scripted questions related to certain steps of the proof (with the most critical being related to all even numbers being divisible by 2 and all odd numbers being *n*+1, where *n* is even). This activity is ambitious and requires students to be proficient at factoring by greatest common factor; if this is a skill that your classes aren't yet proficient in, this discussion and activity should wait until after you have reviewed factoring on Day 4. To evaluate whether this activity should be implemented on Day 3 or Day 4, teachers should include a factoring by greatest common factor question in the "preview" section of the Do Now on Day 1, 2, or 3 and assess whether the prerequisite knowledge can be quickly be filled in there or if more intense remediation on Day 5's factoring lesson is needed before facilitating these proofs.

Day 4 begins factoring polynomial expressions as students have seen them in Math 1 and Math 2. This includes factoring expressions with a greatest common factor (GCF), quadratic trinomials, and the difference of squares. Typically, the difference of squares and GCF methods of factoring can be procedurally taught with little difficulty; however, verbiage surrounding factoring quadratic trinomials can dramatically including "reverse box method," "factoring by grouping," or "divide and slide." For quadratic trinomials, I typically change my instruction year-to-year so that it vertically aligns to what Math 1 and Math 3 have taught the previous year or two in order to activate prior knowledge as much as possible. Due to potential issues with misaligned verbiage with prior instruction, my unit avoids naming steps to a procedure for factoring; I would instead recommend teachers using my unit to determine their instruction based on vertical alignment at their own school. Where Day 4 can begin to integrate elements of philosophy is with Classical applications in geometry that draw from Greek and Islamic conceptions of number and quantity. Rather than deal in abstract expressions (which modern mathematicians have adopted for procedural efficiency), Classical mathematicians dealt with factoring in purely geometric terms. Euclid's Elements outlines the geometric proofs for factoring by GCF (Book II, Proposition 2) and quadratic trinomial (Book II, Proposition 3). Whether teachers opt to include this as a short aside and application questions on practice or center an activity around factoring in geometric terms depends on the needs and current abilities of their classes; however, geometric connections can make polynomial functions feel less abstract in the same way that connecting their operations to numeric functions was done on Day 1.

Day 5 re-establishes a critical relationship for Math 3 between the linear factors of a polynomial expression and its zeros. In the framework of a modern mathematician, this is a relatively straightforward connection to make using order of operations and the zero product of multiplication. Polynomial equations are solved by linear factorization, then solving for each of the factors when set equal to zero; for instance, $(2x-3)(x+5)=0$ would have zeros at $x=\{-5,\frac{3}{2}\}$. Using order of operations, we can see that the sum of 2*x* and -3 when $x=\frac{3}{2}$ or *x* and 5 when *x* = -5 would be zero and resolved before taking the product of the two linear factors, thus the equation would be equal at 0 = 0. A common misconception among students is that linear factors and zeros are one and the same; however, taking time to procedurally check and explain solutions to polynomials in factored form will help to address this issue. Additionally, this begins to reinforce the definition of a function's zero (literally, the value(s) of *x* that make the equation's value zero) and can begin to tie to related and important alternate definitions including solutions (to an equation), roots, and x-intercepts. To remove some of the abstraction to solving polynomial equations using factoring, I would encourage teachers to talk through and discuss quadratics as solved by Al-Khwarizmi (See Appendix 5: Islamic Mathematics) before having students attempt to solve a question that would be relatively straightforward by modern standards using Al-Khwarizmi's methodology.

Day 6 covers factoring cubic expressions and is taught through simple, direct instruction in guided notes, followed by independent practice to allow students time to see a mix of factoring and solving questions. This will give them time and practice with distinguishing what questions are asking (factoring vs solving), as well as the appropriate method for writing the linear factorization of a polynomial. Day 7 is left as a flexible day for teachers to spend additional time and review on the first six days of the unit, to give a short assessment, or to spend time with an extension activity or discussion.

Day 8 utilizes problem-based learning for students to write the standard form equation of a polynomial function given its real zeros, their multiplicity, and the function's end behavior. Students should have prior experience with similar questions writing quadratic polynomials (2^{nd} degree) from their zeros and end behavior in Math 2; however, to this point they have not been asked to connect the same concept to higher-order polynomials (3^{rd} degree and higher), including multiplicity of zeros. I love to think of writing polynomial equations as simply the opposite question that students covered three days prior on Day 5 when they connected the factored form of an equation with solving for its zeros; now students use the zeros to write the factored form of the polynomial, which they will multiply out. In order to prime students to be thinking about the appropriate relationships prior to presenting them with a question aligned to the Math 3 EOC Final Exam, teachers should recap multiplicity, as well as the relationship between linear factors and zeros of a polynomial in either the Do Now or a short review activity afterwards (such as a Kahoot or Quizizz). Having questions scripted to encourage students to consider various aspects of the process for writing polynomial functions is also helpful to successfully facilitating this activity. While the procedural steps can be shown fairly easily with direct instruction, the guided notes should be used as a summary to discussion so that results may be correctly synthesized and easily re-visited in the future.

Days 9 and 10 explore the Fundamental Theorem of Algebra and give time for exercises with applying its results. Unfortunately, there isn't readily a way to formally prove the Fundamental Theorem of Algebra without topology; however, the pattern loans itself well to both discovery and discussion if students are looking for and thinking about the right prerequisite knowledge from the days prior. Key terms for students to have in mind are degree, multiplicity, zeros (frame real zeros as x-intercepts and complex zeros as just that; students should be familiar with this concept from Math 2), and number of turns. This can be quickly reviewed with a mini-lesson, in the Do Now, or as a short assignment like a Kahoot or Quizizz. My discovery activity frames two aspects of the fundamental theorem of algebra: first, students should have a relatively easy time seeing that the number of total zeros is equal to a polynomial's degree through graphical and procedural prompts. Seeing that there can be only an even number of complex zeros (or that they come in complex conjugates) is a trickier jump and may require some scripted questions prepared to help guide discussion. I personally like to ask: “Is it possible to draw a cubic function with an odd number of imaginary zeros?” to frame this second point in discussion. It is impossible to draw a cubic function without real zeros (leaving three complex zeros), and while a cubic function with two x-intercepts appears to have one complex zero, some students should notice that one of the zeros is a double-root that bounces off the x-intercept from the multiplicity activity on Day 2, thereby showing that complex zeros must come in pairs. Because understanding and applying the fundamental theorem of algebra takes time, I have allotted a second day to allow for application exercises about the number and nature of a polynomial’s zeros; this allows for ample discussion and justification, so the teacher should facilitate the exercises in such a way that allows for discussion.

Day 11 synthesizes student knowledge of polynomial graphs and equations by connecting the shape of a polynomial graph to the shape of a roller coaster. I typically spend one day facilitating students starting the activity but allow a 2-3 days for them to finish independently before collecting and grading their assignments. While this assignment certainly is valuable for connecting a phenomenon most students have experience with to polynomial vocabulary, features, and operations, not every teacher may want to utilize this project, in which case they may replace it with a third flex day in the unit for additional review or remediation to address data needs from prior units.

Day 12 covers polynomial long division and builds off the groundwork set on Day 1, connecting numeric addition, subtraction, and multiplication to the same operations with polynomials. I typically teach polynomial long division entirely through direct instruction and for students that already are fluent in numeric long division and operations with polynomials (addition, subtraction, and multiplication), this process is typically picked up pretty quickly although it is tedious, time-intensive, and mistake-prone. For this reason, I typically find it worthwhile to spend 5-10 minutes including numeric long division in my guided notes since the steps translate almost seamlessly to polynomial long division. One especially important part of a quotient to emphasize this day (both with numeric and polynomial division) is the remainder, especially the result that if the remainder of a quotient is zero, then the divisor must be a factor. This relationship will be crucial for the aptly-named Remainder Theorem covered on Days 14 and 15.

Day 13 can take advantage of an opportunity for discovery activity with synthetic division. Synthetic division is a shortcut method of dividing polynomial that only works with a linear binomial as the divisor. With the discovery activity, students will find a quotient using polynomial long division before comparing the same problem to a pre-completed using synthetic division. Seeing the two side-by-side, the goal will be for students to describe the shortcut before summarizing the process using guided notes. Procedurally, this is a fairly straightforward and easy process both to teach and to learn, so a teacher may opt to skip the comparison piece and use only direct instruction so that more time can be spent reinforcing vocabulary, specifically surrounding the linear factor divisor, as well as the remainder (and whether it implies that the divisor is a factor or not).

Day 14 moves from the warm-up into a discovery activity that both practices synthetic division and leads students to naming the remainder theorem. In the activity, students will alternate between finding the remainder of and evaluating for corresponding pairs of problems according to the remainder theorem, given $\frac{p(x)}{x-a} = q(x) + \frac{R}{x-a}$, $p(a) = R$. In my experience, typically the most difficult part of this activity is summarizing the activity. While students usually are close to precisely naming the relationship, it is exceedingly difficult to do so without proper mathematical vocabulary and care must be taken by the teacher to steer all discussion towards the terms outlined throughout the unit. Unlike the fundamental theorem of algebra, the remainder theorem can be proven fairly elegantly using direct proof with relatively straightforward methods. While there is certainly enough on Day 14 for students to see the relationship between evaluating a polynomial and dividing a polynomial, having the proof ready can certainly add to discussion on the relationship and disprove any doubt that the teacher chose selected examples. More importantly, from this day comes several example problems in the summary and guided notes that put the onus on students to think of applying this relationship on whether determining if a linear binomial is a factor of a polynomial, determining whether a value of x is a zero of a polynomial, as well as finding the value of an unknown coefficient in a polynomial expression.

Day 15 allows for more practice with the remainder theorem and its various applications (which can also be used to review polynomial division). Day 16 is allotted for review for the test; Day 17 is the final day of the unit and is given in its entirety for finishing the polynomials test.

Appendix 1: Teaching Standards

- NC.M3.A-SSE.2: Use the structure of an expression to identify ways to write equivalent expressions
- NC.M3.F-IF.7: Analyze piecewise, absolute value, polynomials, exponential, rational, and trigonometric functions (sine and cosine) using different representations to show key features of the graph, by hand in simple cases and using technology for more complicated cases, including: domain and range; intercepts; intervals where the function is increasing, decreasing, positive, or negative; rate of change; relative maximums and minimums; symmetries; end behavior; period; and discontinuities.
- NC.M3.A-APR.2: Understand and apply the remainder theorem.
- NC.M3.A-APR.3: Understand the relationship among factors of a polynomial expression, the solutions of a polynomial equation and the zeros of a polynomial function.
- NC.M3.A-APR.6: Rewrite simple rational expressions in different forms; write $\frac{a(x)}{b(x)}$ in the form (x) + $\frac{r(x)}{b(x)}$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$.
- NC.M3.N-CN.9: Use the Fundamental Theorem of Algebra to determine the number and potential types of solutions for polynomial functions
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Appendix 2: Curriculum Cycle Plan and Resources

Day	Standard(s)	Notes
1	Intro to Polynomials, Operations w/ function notation PK	Do Now: • Review: • Preview Intro to polynomials: • Vocabulary: ○ Standard form (terms, etc.) ○ Factored form (linear factors) • Polynomials as the product of linear factors • Degree (include even and odd functions) Operations with polynomials (function notation) • Connect with Numeric Operations • Addition, subtraction, multiplication
2	Key features of polynomials NC.M3.F-IF.7, NC.M3.F-IF.9	Do Now: • Review: • Preview Direct Instruction: Guided Notes (Part 1) • Degree/even/odd, end behavior, lead coefficient, degree, # of turns, maximae/minimae Discovery Activity • Lead coefficient, degree, number of turns, and end behavior Discussion & Summary: Guided Notes (Part 2) Practice
3	Key features of polynomials	Do Now: • Review

	NC.M3.F-IF.7, NC.M3.F-IF.9	• Preview: Formal Logic Exercise: Review of Polynomial Multiplication (See Appendix 4) • Prove the product of two even functions is even. • Prove the product of two odd functions is odd. Independent Practice: Key Features of Polynomials
4	Factoring quadratics (including difference of perfect squares) A-APR.3	Prerequisite knowledge from Math 1 & Math 2; effectively used to scaffold Unit 4 (Rational Functions) Math History: Euclid's Elements • Book II, Proposition 2 (GCF) • Book II, Proposition 3 (Trinomials) Direct Instruction: Guided Notes Independent Practice
5	Factoring and solving quadratics A-APR.3	Do Now: • Review • Preview Math History: Al-Khwazarmi's Problem (See Appendix 5) Discussion: Zero Product of Multiplication and Factors Guided Notes: Summary of solving polynomial equations

		Independent Practice
6	Factoring and solving cubics (grouping, sum or difference of perfect cubes) A-APR.3	Do Now: • Preview • Review Direct Instruction: Guided Notes Independent Practice: All factoring
7	Unit 3 Flex*	Teachers can use this day for make-up work, relooping material from days 1-6, or addressing data needs from precious units.
8	Relationships b/n factors, zeros A-APR.3	Do Now: • Preview • Review Problem-Based Learning: How can we write the standard form equation of a polynomial given its, real zeros, multiplicity, and end behavior? Discussion, Synthesis, and Summary: Guided Notes
9	Fundamental Theorem of Algebra N-CN.9	Do Now: • Preview • Review Review: Number of Turns, Multiplicity, Degree, and zeros (both real and imaginary). Discussion: Is it possible to draw a cubic function with only one imaginary zero?

		Discovery Activity: Number of Zeros and degree. • Part 1: Degree & x-intercepts • Part 2: Imaginary zeros come in pairs Summary Discussion & Guided Notes
10	Fundamental Theorem of Algebra N-CN.9	Do Now: • Preview • Review Independent Practice: FTA Application questions Discussion of FTA Application Questions
11	Relationships b/n factors, zeros A-APR.3	Do Now: • Preview • Review Application: Roller Coasters Project (Courtesy of Paige Laurain)
12	Polynomial long division A-APR.6	Do Now: • Preview • Review Direct Instruction: Guided Notes • Connect to Numeric Long Division Independent Practice
13	Synthetic division	Do Now:

	A-APR.6	• Preview • Review Discovery Activity: Comparison of Synthetic Division and Long Division. • Compare worked-out examples of each to discover the pattern for synthetic division Summary: Guided Notes Independent Practice
14	Remainder and factor theorem A-APR.2	Do Now: • Preview: • Review Discovery Activity: Dividing & Evaluating Polynomials • Connect that p(x)x-a=q(x)+r and p(a)=r Summary & Discussion: Guided Notes • Describing the remainder theorem • Uses & applications of the remainder theorem
15	Remainder and factor theorem A-APR.2	Do Now: • Preview: • Review Discussion: Connections between polynomial division, the remainder theorem, and zeros. Practice Exercises to review polynomial division and the remainder theorem,

15	Unit Review Day PK	
16	Unit 3 Test	

Appendix 3: Connecting Polynomials with Numeric Operations

As de Groot and Boyajian suggest, connecting polynomial operations with the numeric operations students are familiar with dating back to elementary school can help students see structure in expressions by drawing parallels with prior knowledge. This not only builds fluency, but doesn't present potential barriers to future learning by leaning entirely on layman's analogy in lieu of utilizing proper mathematical vocabulary.

To connect polynomial expressions to numeric expressions, students should revisit the concept of base ten places. For instance, the number 547 may be written as such:

$$547$$

$$500 + 40 + 7$$

$$5 \cdot 100 + 4 \cdot 10 + 7$$

$$5 \cdot 10^2 + 4 \cdot 10 + 7$$

Seeing a numeric expression written in terms of ones, tens, and hundreds should be able to lead to discussion of expressions from grade school, such as "we don't combine ones and tens" For instance:

$$547 + 251 = 798$$

$$(5 \cdot 10^2 + 4 \cdot 10 + 7) + (2 \cdot 10^2 + 5 \cdot 10 + 1) = 7 \cdot 10^2 + 9 \cdot 10 + 8$$

With the new format of numeric expressions, students should be quite eager to make the connection to addition of polynomials and the rule that only like terms may be combined:

$$(5x^2 + 4x + 7) + (2x^2 + 5x + 1) = 7x^2 + 9x + 8$$

Depending on students' elementary and middle school experiences, they may have learned the FOIL (first, outer, inner, last) method of multiplication for both double-digit numeric expressions and linear binomials. While this method does allow students to see structure in the sense that the structures of polynomials are connected to existing knowledge as recommended by de Groot and Boyajian, there are some shortcomings to FOIL. As pointed out by NCTM contributor, Isaac Frank, FOIL becomes impractical for multiplication with higher-order polynomials and, as such, the "Box Method" of multiplication highlights connections to factoring by grouping without sacrificing, if not improving, procedural efficiency[18]. Most secondary math teachers are likely familiar with the box method, which lists the terms of each polynomial factor as the rows and columns, allowing a mathematician to quickly distribute all terms before summing like terms:

$$(3x^2 - 4x + 5) \cdot (2x - 1)$$

	$3x^2$	$-4x$	5
$2x$	$6x^3$	$-8x^2$	$10x$
-1	$-3x^2$	$4x$	-5

$$6x^3 - 3x^2 - 8x^2 + 4x + 10x - 5$$

$$6x^3 - 12x^2 + 14x - 5$$

Making a similar connection to the places of base-10 number systems in the manner that de Groot and Boyajian make, the box method can similarly be adapted for numeric multiplication with any number of digits:

$$129 \cdot 34$$

	100	20	9
30	3,000	600	270
4	400	80	36

3,000+400+600+270+80+36

4,386

Appendix 4: Even/Odd Function Proofs

The crux of proving that the product of two even functions is even or the product of two odd functions is even lies in proving the sum of two even numbers is even and that the sum of two off numbers is even. As such, I will refer to two lemma (helper proofs) for teachers to quickly reference before our main proofs regarding polynomial functions.

Lemma 1: The sum of two even numbers is even.

Assume there are two discrete, even integers *x* and *y*.

Because *x* and *y* are even integers, both numbers are divisible by 2. (Definition of an even number).

Thus, it can be shown:

x + *y* = *z*, where *z* is an integer (addition of integers is closed)

[18] Frank, Isaac. “When FOIL Falls Apart.” *The Mathematics teacher*

$2(p + q) = z$, where p and q are both integers (distributive property of multiplication; definition of an even number).

Since $2(p + q)$ is divisible by 2, z must also be divisible by 2 (transitive property), therefore the sum of two even integers is even (definition of an even number).

Lemma 2: The sum of two odd numbers is even.

Assume there are two, discrete odd integers, p and q.

Since p and q are odd numbers, they may be re-written as $m + 1$ and $n + 1$, respectively, where m and n are discrete even integers. (Definition of an odd number).

Thus, it can be shown:

$p + q = a$, where a is an integer (addition of integers is closed)

$(m + 1) + (n + 1) = a$ (transitive property)

$m + n + 2 = a$ (addition of like terms)

$2(x + y + 1) = a$, where x and y are integers (definition of an even number; distributive property)

Since $2(x + y + 1)$ is divisible by 2, $p + q$ must also be divisible by 2 (transitive property), therefore the sum of two odd integers must be even (definition of an even number).

With these two lemma, it is relatively elementary to show the product of two arbitrary polynomial functions using the rules of polynomial multiplication.

Theorem 1: The product of two even functions is even.

Assume two even functions, $f(x) = ax^m + \cdots + c$ and $g(x) = bx^n + \cdots + k$.

If $f(x)$ and $g(x)$ are even functions, then their degrees m and n must also be even (definition of an even function).

Thus, it can be shown:

$(f \cdot g)(x) = (ax^m + \cdots + c) \cdot (bx^n + \cdots + k)$

$(f \cdot g)(x) = a \cdot bx^{m+n} + \cdots + c \cdot k$ (multiplication of a polynomial function)

The resulting product of $f(x)$ and $g(x)$ is an $m + n$ degree polynomial (definition of degree)

The sum of two even numbers is even (lemma 1), therefore, the product of even polynomials $f(x)$ and $g(x)$ is even.

Theorem 2: The product of two odd functions is even.

Assume two odd functions, $f(x) = ax^m + \cdots + c$ and $g(x) = bx^n + \cdots + k$.

If *f(x)* and *g(x)* are even functions, then their degrees *m* and *n* must also be odd (definition of an odd function).

Thus, it can be shown:

$(f \cdot g)(x) = (ax^m + \cdots + c) \cdot (bx^n + \cdots + k)$

$(f \cdot g)(x) = a \cdot bx^{m+n} + \cdots + c \cdot k$ (multiplication of a polynomial function)

The resulting product of *f(x)* and *g(x)* is an $m + n$ degree polynomial (definition of degree)

The sum of two odd numbers is even (lemma 2), therefore, the product of odd polynomials *f(x)* and *g(x)* is even.

Appendix 5: Classical Islamic Mathematics (courtesy of Steven Krantz)

For modern mathematicians, quantities are abstracted for procedural efficiency and ease of relation to multiple representations that allow for more complicated analysis and application. Understanding Classical mathematics used by Greek and Arab mathematicians, however, provides a more concrete framework for representing polynomial equations.

Consider the relatively straightforward (by modern standards) equation[19]:

$$x^2 + 10x = 39 \qquad \text{or} \qquad x^2 + 10x - 39 = 0\,.$$

Naturally, the quadratic formula can be used to quickly find the solutions to this equation,

$$x = \frac{-10 \pm \sqrt{10^2 - 4 \cdot (-39) \cdot 1}}{2} = \frac{-10 \pm \sqrt{256}}{2} = \frac{-10 \pm 16}{2}\,.$$

This yields two solutions, -3 and -13.

Arab mathematicians of Al-Khwarizmi's time did not have negative numbers, and thus only the positive solution would be taken; the side of a square:

$$x = \frac{-10 + 16}{2} = 3\,.$$

Now, to conceive of the equation as Al-Khwarizmi saw it, the equation would be need to shown as the sum of quadrilaterals' area. x^2 would be shown as a square with *x* by *x* dimensions, where

[19] Krantz, Steven G. *An Episodic History of Mathematics : Mathematical Culture through Problem Solving,* 2010. p 60-61

10*x* would be represented as four rectangles of dimensions 2.5 by *x*, such that their areas would be 2.5*x* yielding a sum of 10*x*:

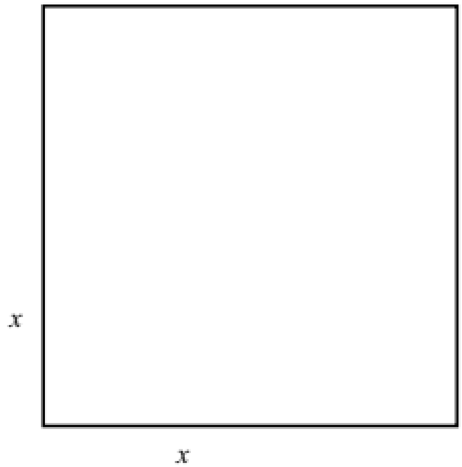

Figure 4.1. A problem of Al-Khwarizmi.

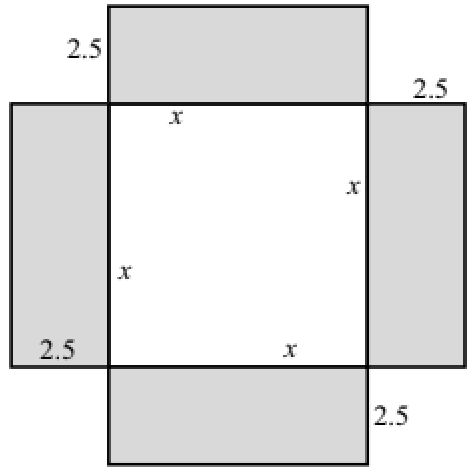

Figure 4.2. Sum of shaded areas is $10 \times x$.

Now according to the original problem, $x^2 + 10x = 39$, we know that the sum of the areas in Figure 4.2 must be 39. This would be handled by filling in the areas of the corners with squares to create a perfect square, as shown in Figure 4.3:

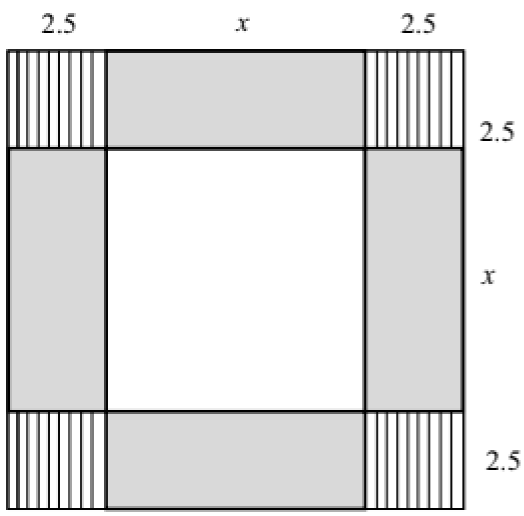

Figure 4.3. Area of large, inclusive square is 64.

Each of the squares is of dimensions 2.5 by 2.5, giving the large square an area of

$$39 + 2.5^2 + 2.5^2 + 2.5^2 + 2.5^2 = 64$$

Since the large square has area 64, it follows each of its side lengths must be 8. Knowing that the length of each side is x+2.5+2.5, it can be shown $x + 2.5 + 2.5 = 8$; thus x=3.

<u>Appendix 6: Direct Proof of the Remainder Theorem</u>

Given the quotient $\frac{p(x)}{x-a} = q(x) + \frac{R}{x-a}$, where *p(x)* and *q(x)* are polynomial equations, *a* is an integer, and *R* is a real number, it can be shown:

$\frac{p(x)}{x-a} = q(x) + \frac{R}{x-a}$,

$\frac{p(x)}{x-a} = q(x) \cdot 1 + \frac{R}{x-a}$ (identity property of multiplication)

$\frac{p(x)}{x-a} = \frac{q(x)(x-a)}{1(x-a)} + \frac{R}{x-a}$, (division of a number by itself is 1)

$\frac{p(x)}{x-a} = \frac{q(x)(x-a)+R}{(x-a)}$, (addition of fractions)

$(x-a)\left(\frac{p(x)}{x-a}\right) = (x-a)\left(\frac{q(x)(x-a)+R}{(x-a)}\right)$, (multiplying both expressions in an equality by the same factor preserves equality)

$p(x) = q(x)(x-a) + R$ (multiplication is the inverse of division)

Now let x=a:

$p(a) = q(a)(a-a) + R$ (substitution)

$p(a) = q(a) \cdot 0 + R$ (subtraction of a number from itself is zero)

$p(a) = 0 + R$ (zero property of multiplication)

$p(a) = R$

Appendix 7: A Graphic Organizer for Polynomial Properties

In her NCTM article, Donna M. Young discusses a trapezoid-shaped organizer intended to connect several key properties of polynomial expressions, including the degree of a polynomial, solution of a system of equations, the zeros of a polynomial expression, standard form, and factored form. In doing so, students can begin to view polynomial properties as a coherent concept as opposed to a series of unrelated exercises[20].

[20] Young, Donna, 61-2.

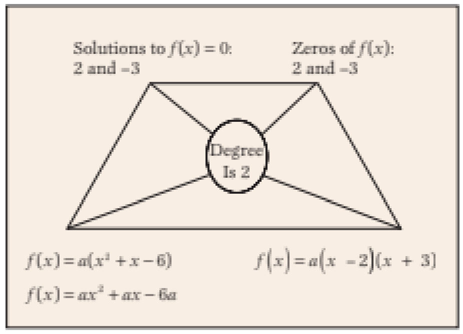

Fig. 1 The isosceles trapezoid is an ideal shape for the graphic organizer.

I would add that a sixth element of x-intercepts on a pentagon-shaped organizer would be helpful in the context of Math 3, where the EOC assessment is entirely calculator-active. As such, understanding the relationship between algebraic and graphical representations of polynomials is a considerable advantage to students as they may draw on utilizing technology for a number of problems that may otherwise require tedious, time-consuming, and mistake-prone methods to solve.

Bibliography:

Irving, Ron. *Beyond the Quadratic Formula*. Washington, D.C: Mathematical Association of America, 2013.

Krantz, Steven G. *An Episodic History of Mathematics : Mathematical Culture through Problem Solving* Washington, DC: Mathematical Association of America, 2010.

Cornelis de Groot, and Meredith Boyajian. "Like What's in a Terms: Name?" *The Mathematics teacher* 108, no. 7 (2015): 506–.

Young, Donna M. "A Graphic Organizer for Polynomial Functions." *The Mathematics teacher* 106, no. 2 (2012): 160–162.

Frank, Isaac. "When FOIL Falls Apart." *The Mathematics teacher* 112, no. 6 (2019): 460–461.

Weiss, Michael. "Factor and Remainder Theorems: An Appreciation." *The Mathematics teacher* 110, no. 2 (2016): 153–156.

Laurain, Paige. "Roller Coaster Engineering Project." Unpublished Document, courtesy of author. May 2019.
https://docs.google.com/document/d/13eZarMIAiGbqOvOa8hiZ_88iM815z63I/copy

Lone, Jana Mohr and Michael Burroughs. *Philosophy in Education: Questioning and Dialogue in Schools*. Lanham, Maryland: Rowman & Littlefield, 2016.

Dewey, John, 1859-1952. 1966. *Democracy and Education: An Introduction to the Philosophy of Education*. New York: The Free Press.