Time Manipulation:
A How-To Guide

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This curriculum unit is recommended for:
IB Math SL1, Precalculus, or AP Calculus AB

Keywords: limit, continuity, intermediate value theorem, IVT, infinity

Teaching Standards: See Appendix 1 for teaching standards addressed in this unit.

Synopsis: Sometimes, seeing the entirety of a function is useful. Sometimes, it can be overwhelming. In this unit, students will learn to manipulate time to focus on the important parts of a function. Do they need to see what is happening at a specific time? They can slow everything down and zoom in on that instant! Do they need to see what is happening at the far ends of the function? Let's speed up time and zoom to infinity! In this unit, students will use their new-found time manipulation powers to find the limits of functions which will allow them to determine if functions are continuous and then to apply the intermediate value theorem. Students will start the unit with a debate about what "close" means and why infinitesimals might not actually exist and will end up climbing over desks and trying to escape from the classroom without using the door.

I plan to teach this unit during the coming year to 70 students in IB Math SL1.

Introduction

"Get closer!" repeats the photographer at our annual back-to-school staff picture. We dutifully squeeze in, trying to get this all over with. "Closer!" he calls again. Shoulders start touching; bodies begin angling like teeth in an overcrowded mouth. Frustratingly, the photographer roars "closer!" yet again. We exhale, roll our eyes, and our touching shoulders become touching arms, wrists, hands, and emotions. We cross our fingers (figuratively – there is no space to do it literally) and we hope that the photographer is satisfied. What were once minutes have started to ooze into seconds, the hands of the clock melting and dripping off of its face like the sweat off of our own. Through this slow and sticky haze of time, the photographer's mouth opens in slow motion. We're restless and hopeless - trapped by time and by space. There's nowhere to go but— "Closer!"

What counts as close? Is there an ultimate "closeness" in which two numbers or objects cannot possibly get any closer to each other? From the photographer's point of view, people can always get closer to each other. However, as people get closer and closer, time starts to slow down. Every second takes longer and longer until time almost stands still, much like the compressed teachers. As it turns out, a similar thing happens in math when numbers get close to each other. It is easy to name a number close to, say, 87. Eighty-six is pretty close but, then again, 86.5 is closer. Actually, 86.75 is closer. Better yet, 86.999999999999 is really close to 87 but, no matter how close of a number we name, there's always a closer number – we just have to add more decimal places. Adding more decimals means that it takes longer and longer to name a close number and thus it feels like time is slowing down again.

I have spent a lot of time thinking about the abstract idea of close numbers (which are called limits) because it is foundational to understanding higher-level mathematics. Everything in calculus and beyond relies on understanding the idea of closeness and the infinitesimal (which is the difference between two numbers that are close). Throughout my teaching career, I have noticed that students really struggle with the inexactness of "close" and, more than that, they struggle to even see the point of closeness in math. Why would you plug "almost 4" into an equation? Why not just 4? Unfortunately, there are many students who cannot get past these struggles and who are thus unable to fully access higher-level mathematics. The goal of this curriculum unit is to aid those students in developing a better understanding of closeness (i.e. limits) and infinitesimals so that they can move from arithmetical math into abstract math.

My students are usually successful in math, with the vast majority earning A's or B's in all of their previous math classes. Some of them are good at memorizing processes and formulas and others are good at understanding and making connections. The ones who are good at memorizing are starting to find that it is not enough anymore and their grades are starting to slip down below the level with which they are comfortable. All of my students are in the IB program and most have the goal of getting an IB diploma. They are college-bound, taking a full load of IB courses, involved in tons of activities, and, in general, they are overwhelmed. Several of my students come with accommodations for ADHD and/or anxiety. In spite of all of that, they are generally curious, good-natured, and willing to do whatever I ask of them, especially if they know why I am asking it. The problem with this curriculum unit is that my students do not know why we need to do it and, on top of that, they also do not understand why they struggle with this concept more than any other one we have explored so far. Generally, I take a very abstract approach to teaching limits because that is how I understand the topic. My students clearly understand things differently than I do and so I need to take a more concrete approach that connects to their own personal experiences.

The student population at our school is diverse, with 44% of the students being African American, 22% Hispanic, 21% white, 9% Asian, and 4% being more than one race. Fifty-two percent of our student body is female and 48% is male. We are a Title I school with approximately 2000 students and 58% of our students receive free or reduced price lunches.[1] In my IB Math SL1 classes, 58% of my students are female and 42% are male. My classes are still diverse but white and Asian students are over-represented (37% and 12% respectively) while African American students are under-represented (21%). I have three different sections of IB Math SL1; one section has 30 students, another has 20, and the third has 17. My classes are 90 minutes long and meet every other day. My students are all 10th or 11th graders and, after they complete this course, they will either take IB Math SL2 or AP Calculus AB/BC.

For this curriculum unit, the learning goals are as follows:

- Students will understand the informal ideas of an infinitesimal, a limit, convergence, and continuity
- Students will be able to communicate with limit notation for one- and two-sided limits
- Students will be able to use graphs, tables, and algebra to calculate finite and infinite limits

In general, my goal for this unit is to get students comfortable with the idea of getting infinitely close to a number without actually reaching it. They should also be able to differentiate between being close to a point and actually being at a point. I would also like for them to understand why those two concepts are important and in what situations each of them are useful. Previously, I have not included any real world problems in this unit but I would like to have students also apply limits to situations that are not strictly abstract. Prior to this unit, students will have studied all of the basic parent functions with the exception of the trigonometric functions. Students will know how to simplify various algebraic expressions including exponential and logarithmic expressions. Students will also be able to graph functions, including piece-wise functions, with and without a calculator. They will also be able to interpret and evaluate functions in a variety of representations, including tables, graphs, algebra, and words. In addition, students will also have studied sequences and series, particularly arithmetic and geometric sequences and series. They will already know what "convergent" and "divergent" mean in a mathematical context as well as being able to determine if a series converges. Until now, though, everything they have done in math has a precise answer. "Almost" and "close to" have not been part of the solving problems – at least, not since they learned about estimating in elementary school. This unit changes all of that.

My students are not the first group of people to have a problem with the idea of infinitesimals. "Even as late as the 1730s the High Church Anglican bishop George Berkeley mocked mathematicians for their use of infinitesimals, calling these mathematical objects 'the ghosts of departed quantities.'"[2] Infinitesimals have long been controversial. Everything else in math is quantifiable, even the so-called imaginary numbers. How is it possible that mathematicians are completely fine with something as ill-defined as the infinitesimal? And make no mistake – it is ill-defined. The Stewart book, a gold standard amongst calculus textbooks, defines an infinitesimal as "arbitrarily small" or "sufficiently small (but not 0)."[3] In fact, by its very nature, the infinitesimal cannot be defined. As shown above, with the photographer example, it is always possible to move closer to an object or number. If we decided to define the infinitesimal as being equal to 0.1, then we are saying that it is not possible to be any closer to another number than that which is easily disprovable. In fact, no matter how small we decided to make the infinitesimal, we could always choose a smaller number by adding in another zero. In other words, instead of the infinitesimal being 0.1, we could add a zero and make it 0.01. We can still go smaller though: 0.001 or 0.0000001. There is no limit to how many zeroes we can include and thus there is no possible way to exactly define what an infinitesimal equals.

Content Research

Amir Alexander succinctly describes the problem with defining infinitesimals, which he refers to as "indivisibles," in his book, *Infinitesimal*:

> Every line is composed of a string of points, or "indivisibles," which are the line's building blocks, and which cannot themselves be divided. This seems intuitively plausible, but it also leaves much unanswered. For instance, if a line is composed of indivisibles, how many and how big are they? One possibility is that there is a very large number of such points in a line, say a billion billion indivisibles. In that case, the size of each indivisible is a billion-billionth of the original line, which is indeed a very small magnitude. The problem is that any positive magnitude, even a very small one, can always be divided…The other possibility is that there is not a "very large number" of indivisibles in a line, but actually an infinite number of them. But if each of these indivisibles has a positive magnitude, then an infinite number of them arranged side by side would be infinite in length, which goes against our assumption that the original line is finite. So we must conclude that the indivisibles have no positive magnitude or, in other words, that their size is zero. Unfortunately, as we know, $0 + 0 = 0$, which means that no matter how many indivisibles of size zero we add up, the combined magnitude will still be zero and will never add up to the length of the original line. So, once again, our supposition that the continuous line is composed of indivisibles leads to a contradiction.[4]

So then, if an infinitesimal cannot have a positive size or zero size, what can it be? (The obvious – but completely wrong – answer that presents itself is "it has a negative magnitude!" That idea has all of the same problems as an infinitesimal with a positive magnitude though since negative numbers can be divided just as easily as positive ones. It also has the additional problem of making no sense, as the distance between two numbers cannot be negative.) If the idea of an indivisible mathematical building block is paradoxical, than perhaps it does not actually exist at all. In other words, infinitesimals do not exist precisely because quantities can always be split into smaller and smaller pieces. In fact, both Plato and Aristotle agreed that "the concept of infinitesimals was erroneous and that continuous magnitudes can be divided ad infinitum."[5] At that time (300-400 BC), geometry was the king of mathematical reasoning. If something could be proven geometrically, it became an unalienable truth of mathematics. If it could not be proven geometrically or worse, if it led to a geometric paradox like the infinitesimal did, then the idea was disregarded. Therefore, in the world of Plato and Aristotle, an infinitesimal did not pass mathematical muster and it made sense for them to discard the idea. A century later, however, Archimedes once again examined the infinitesimal and "fully aware of the mathematical risks he was taking, he chose to ignore the paradoxes of the infinitely small thereby showing just how powerful a mathematical tool the concept could be."[6] Like the imaginary number, the infinitesimal has become the foundations of modern math. Without it, calculus would not exist. Archimedes had an inkling of the powerful things that could come from the infinitesimal but, unfortunately, he did not have any students to carry on his work and other mathematicians avoided his un-geometric ideas. For millennia, the existence of the infinitesimal was kept in the cobwebbed corners of history, sometimes examined but often forgotten and ignored.

In the 1500s, a group of progressive mathematicians reexamined the idea of the infinitesimal and deciding that its usefulness outweighed its risks and indefinability, they decided to incorporate it into their work. However, the Jesuits had a huge problem with this. "We consider this proposition to be not only repugnant to the common doctrine of Aristotle, but that it is by itself improbable and is disapproved and forbidden in our Society,"[7] ruled the Jesuits. "The vision of eternal order was, to the Jesuits, the only reason mathematics should be studied at all. If infinitesimals were to prevail, it seemed to the Jesuits, the eternal and unchallengeable edifice of Euclidean geometry would be replaced by a veritable tower of Babel, a place of strife and discord built on teetering foundations, likely to topple at any moment. The new mathematics undermined the very possibility of universal order."[8] The Jesuits, much like my students, expected math to be immutable and perfect. Math used reasoning and logic and was always true. There were no messy ideas because math exactly described the physical world and that world had been designed by the Creator. Thus the idea of an ill-defined, paradoxical infinitesimal was immediately dismissed. Math could not be disorderly because the Creator was not disordered. The struggle between mathematicians and the Jesuits continued for a century. Ultimately, there is still a struggle today between different factions of mathematicians because of the vaguely defined concept. Even Bertrand Russell, a philosopher and mathematician in the early 1900s, declared that infinitesimals are, at best, a "pseudoconcept."[9] So it's only natural that students struggle with the idea, too.

Once students have made peace with infinitesimals (whether they believe in them or not), the idea of a limit can be explored. Limits look at the value of a function as the input gets close to a certain number. Functions often describe real world situations and, usually, the input (or the independent variable – the one we can only choose but not control) is time. To get arbitrarily close to a certain time, we need to be able to slice time into smaller and smaller pieces, which is actually quite an easy thing to do these days since "time is more accurately measured than any other physical entity. It has become increasingly precise."[10] This precision allows us to break time down into unthinkably tiny pieces. There are a couple of problems, of course. The first is that time, like the infinitesimal, may not actually exist. The second problem is that (assuming time does exist), we have a lot of trouble observing such small durations with our own senses. Things move and change too quickly for us to see with our own eyes. Luckily, the fist problem (the existence of time) actually helps us deal with the second problem.

There is a potential that time does not actually exist, or at least not in the way that we typically think of it. "Nothing in known physics corresponds to the passage of time. Indeed, physicists insist that time doesn't flow at all; it merely is…Physicists prefer to think of time as laid out in its entirety – a timescape, analogous to a landscape – with all past and future events located there together."[11] This makes sense from a mathematical perspective, too; after all, we can see a whole function all at once and we can use any time that we want to as the input for a function. We do not need to wait for time to pass until we get to the input we want – eagerly watching our stopwatch, holding our breath as the time gets closer, and then shouting, "plug in 8 minutes now!" We can just choose any time, put it into our function, and then see what the outcome is. In a sense, it is a way of time traveling but it is traveling in the same way that walking across a room is traveling. Time becomes just another dimension. "All moments, past, present and future, always have existed, always will exist...It is just an illusion we have here on Earth that one moment follows another one, like beads on a string, and that once a moment is gone it is gone forever."[12] An immediate example of that is the paper you're reading right now. The whole paper is here, was here before you started reading and will be here after you're done but you're only reading one sentence of it at a time. You don't know what's in the upcoming sentences because those are in the future. The sentence you're currently reading is the present and the ones you've already read are the past. You can go back and forth between sentences (you can time travel right now!) or you can read them sequentially from the first to the last (if you're a traditionalist). Unfortunately, traveling through time in our real lives is not as easy as skipping around in a paper but, in math and physics, it really is that easy. And so we come back to the idea of limits.

To really be able to accurately assess what a function is doing, we need to slow down time. The superhero Quicksilver demonstrates this perfectly in the movie X-Men: Days of Future Past. A showdown is about to happen – the bad guys face off against the good guys and fire their guns before anyone can react. Well, anyone aside from Quicksilver. He is able to travel faster than the bullets, the rain, Wolverine's claws, and pretty much everything. While he moves around the room at this supersonic speed, he has plenty of time to observe everything that's happening because it all seems to be moving in slow motion relative to him. He moves individual bullets, people's hands, knives, soup, and anything else that he wants to.[13] Just as Quicksilver slows down time by completing more movements in smaller and smaller durations, we can also do that with functions. If we need to see what is happening at a given point (i.e. if we need to find the limit), we can take more measurements, getting closer to that point in increasingly small steps in time. As we start evaluating the functions at smaller intervals, its value becomes clearer and clearer just as, when Quicksilver started moving faster, it became easier and easier to see where the bullets were. At a certain point, if we take small enough steps in time, the function's movement appears to cease and it is obvious what the limit is.

On the other extreme, sometimes we want to figure out what the "ends" of a function are doing. ("Ends" is a common math term but it is slightly misleading because, typically, a function continues infinitely in both directions so functions do not really have ends in the usual sense of the word.) For this, we need to speed up time by taking larger and larger steps. An example of this comes from Star Trek: Voyager. On one of their missions, the Voyager comes across a previously unknown planet. After some exploration, the people on the Voyager spaceship discover that they are traveling through time approximately 85,000 times slower than the people on the planet, Tahal-Meeroj[14] are experiencing it.[15] When the Voyager first arrives, the people below are in their stone age. Within hours on the Voyager, the planet's civilization has advanced to the equivalent of medieval Earth. The crew of the Voyager watches this evolution and the first officer comments, "We might miss the rise and fall of a civilization" to which the chief engineer responds, "So, we'll watch the next one."[16] Later on in the episode, after the people develop space technology, two astronauts travel to the Voyager and one of them asks, "So you really haven't been watching us for centuries?" The captain replies, "Actually, we just got here."[17] This episode parallels what happens when we look for the end behavior of functions. We have to get as close to infinity as we can which means traveling across large amounts of time, which is what the Voyager did relative to the people of Tahal-Meeroj. From the planet's point of view, the Voyager and its inhabitants move incredibly slowly (in space) but it was exactly that slow spatial movement that allows them to travel across so much time so that they could see what happened at the "end" of the planet's timeline. When looking for the limit of a function as its input approaches infinity, we start by choosing a very large number as the input. Then we choose an even larger number to input. We keep repeating that process, choosing larger and larger inputs until the function stabilizes and stops changing. Then, we can make our observation about what the end of the function does. Again, this paper is another example of taking large steps through time. We started examining the situation in 300 BC with no idea how infinitesimals would be connected to time. Then, after moving across large stretches of time to 1500AD, to 1900AD, and finally ending up in the 24th century, we know see how infinitesimals, limits, and time are all related to each other. Time manipulation is a very useful skill to have when you're doing math.

Instructional Implementation

Teaching Strategies

When presenting all of this information to students, I plan to start much the same way that I went through the history of infinitesimals. We will start by discussing what "close" means in various contexts: in elevators, at the school dance, while driving in bad weather, in class during a test, in horseshoes, and so on. We will try to come up with a definition and we will try to figure out how close two things can be in all of those contexts. From there, we will generalize "closeness." I will ask students if there is an ultimate closeness where two things could not possibly be any closer. I also want to introduce them to Zeno's dichotomy paradox by telling them I am going to run into the classroom wall. Naturally, before I run all the way into the wall, I have to run halfway across the room and so I do. Then, before I can run the rest of the way, I have to run half of that distance. And so on until my students are disappointed that I did not run into the wall. (But did I get close to the wall? We will talk about it. They will likely say, "Not close enough.") Another of Zeno's paradoxes is the Arrow Paradox which is very similar to the line and indivisibles example given previously. In any instant of time, an arrow in flight is not moving. You can look at whatever instant you want to, but the arrow will always be still. And yet, the arrow manages to fly across a distance if you look at time as a whole. But, as we saw before, adding a lot of zeros together results in zero so where does that motion come from?

From those conversations and demonstrations, we will move onto the idea of the infinitesimal and possibly try to come up with our own definition. At some point, I expect my students will also split into two factions: those opposed to the infinitesimal and those who support it. I will give them the chance to debate with each other, to point out the problems in the other side's arguments, to ask questions, and to challenge each other's thinking. I do not expect there to be a unanimous agreement about the existence of the infinitesimal (since there is not even agreement amongst current day mathematicians). At best, we will come to the decision that, even if we cannot agree on its existence, we can agree on its importance.

From there, we will examine that importance and what the infinitesimal can actually do. We will revisit Zeno's paradoxes through a new lens, where time and space cannot be infinitely divided but rather are made up of discrete chunks – atoms for space and chronons (or moments) for time. When its impossible for space to be infinitely divided, then Zeno's dichotomy paradox only works until I am one atom away from the wall then there are no more half distances to cover, only one whole indivisible atom. Finally, I will run into the wall and my students will see the utility of the infinitesimal.

Next comes the idea of a limit. Students are familiar with limits in their everyday lives. They can give me all sorts of examples of limits they have to deal with on a daily basis: speed limits, curfews, due dates for assignments, bells signaling the beginning of class, and so on. As teenagers, they are deft at getting as close to a limit as possible without crossing it. They know exactly what the boundaries are and they do not hesitate to run right up to them. In class, students will share examples of limits in their own lives and how close they can get to them. We will also play some guessing games together: what temperature is it in the classroom? How many jellybeans are in the jar? How close can you get to timing 10 seconds without a clock? Students will do their best and the winners will get some sort of prize, even though the games are all just luck. We will connect the games back to infinitesimals, closeness, and limits. Then, we will try to describe those games using mathematical notation. I will lead them from the informal vocabulary to the formal notation for limits.

As we develop the formal notation, we will also start examining functions in their various representations. Unbeknownst to students, all of the games and conversations we have had so far were functions in a verbal form. I will make that connection clear for them and then we will move into functions represented as graphs, then tables, and then algebra. During this time, we will also watch the videos I cited earlier (X-Men and Star Trek) and discuss their relationship to limits. We will also spend a lot of time talking about time itself. As described previously, limits are closely related to time manipulation so students will need to have an understanding of how to slow down/speed up time to properly calculate limits. We will come up with a Time Manipulation how-to guide for themselves and potentially to share with future students to aid them in finding limits. The guide will include not only how to manipulate time but also the situations in which each method would be appropriate.

After that, we will travel back to the beginning of the unit and revisit our discussion about infinitesimals. Students will reflect on the unit and reevaluate their feelings about infinitesimals. We will discuss the importance and existence of infinitesimals and students will resume their debate. Finally, the unit will end with an exam that tests their understanding of limits as well as their skill in using limits to solve problems.

Classroom Lessons and Activities

General Timeline of Unit

Day 1. Closeness and Infinitesimals	Day 2. Intro to Limits and formalization. Graphs without numbers	Day 3. Graphs with Numbers. Tables.	Day 4. Revisit debate. Quiz.	Day 5. Graphs and functions. Finding limits algebraically
Day 6. Finding limits algebraically, continued	Day 7. Continuity	Day 8. Intermediate Value Theorem	Day 9. Review	Day 10. Self-Assessment and Test

Description of Unit and Activities

To begin the unit, students will watch the Quicksilver scene from X-Men, Days of Future Past. We will begin with the real-time version of the scene, in which we see the events from the other characters' perspectives. In this version, we see our heroes in a dire situation when, all of a sudden, the bad guys get knocked out, drop their weapons, and fall unconscious. After watching the video, students will be asked several questions: What happened? How close did Quicksilver get to the knives and to the bad guys? How close did the bad guys get to winning? How close did the good guys gets to losing (or dying)? How close did the bullets get to the good guys? Students will discuss their answers in a very brief Think/Pair/Share because, at this point, they should not really be able to answer any of the questions. Then we will watch the same video but from Quicksilver's point of view. After this viewing, students will again do a Think/Pair/Share to discuss the same questions as before. This time, they should be given more time to discuss the answers since they will actually be able to see what happened. As an extension, students can also be asked to discuss why the second video gave them more answers than the first video and why the videos were in slow-motion or not depended on which characters' perspectives we were following.

After the introductory videos and discussions, students will be put into groups of three to four for a board meeting. The goal of the board meeting is to define the term "close" in a general, real-world way and in a mathematical way. Students should have 5-10 minutes to discuss their definitions within their groups and to write their definitions and examples on their boards. Then, the groups will all come together and the whole class will have a board meeting. The details of the board meeting are described in Appendix 2.

Now that students have a good working definition of "close", we will formalize the definition through lecture and note-taking. The lecture will include the mathematical definition of "close," a mathematical example, as well as introducing the vocabulary term "infinitesimal." Once the students have the idea that closeness can be defined in terms of the infinitesimal, we will have a debate about whether it actually exists or not (which is and has been an issue in mathematics since the possibility of the infinitesimal was first introduced). Students will watch a video on Zeno's Dichotomy Paradox that explains the problem: an arrow is still at any given moment in time and a stretch of time is composed of several moments so how does the arrow move forward? The goal of the video and the conversation surrounding it is not to come to any conclusion or consensus. The goal is to engage the students in a conversation about infinitesimals that is ongoing in the broader mathematical community. There are no right or wrong answers here. After watching the video, students will sort themselves into groups for a debate, the details of which are in Appendix 2. Students will explain and defend their stances about whether the infinitesimal exists (or if it even matters that it exists) and will have an opportunity to change their stances after the debate. (Note: This debate will be revisited throughout the unit, particularly after students have learned more about limits and have more of a context for infinitesimals.)

The next lesson begins with a lot of signs posted around the classroom. (Examples of these signs are in Appendix 2). Students will start with a Gallery Walk during which they will walk around the classroom and leave written comments on the signs about how the students obey (or do not obey) those signs. More detailed Gallery Walk instructions are in Appendix 2. The goal of this activity is to build the idea that limits can mean different things in different contexts. Some limits, like laws or not getting Gremlins wet, are firm: we can get close to doing them but not actually do them. Some limits, like due dates, we can actually reach and, finally, some limits, like not eating before swimming, can be slightly crossed with minimal risk. After the Gallery Walk, the teacher will lead the class in a discussion about things that they noticed or wondered about – what similarities they saw, what differences they saw, and what connections they made. The discussion should also relate to math. Students will be asked questions such as: how does this relate to closeness and infinitesimals? Ideally, students will use infinitesimals or closeness to describe their behavior regarding the different limits.

From that discussion, we can begin to formalize the idea of a limit. Students will work in pairs to go through the Limits Explorations in Appendix 2. Each pair of students could work on all three of the explorations or each pair could focus on one exploration and then share their results with the class. After doing the exploration, there will be a brief lecture on the formal definition and notation of limits (both 2-sided and 1-sided limits). The examples at this point will be graphical (not numeric or algebraic). After the lecture, students will individually work on the Know Your Limits worksheet. Alternately, this worksheet could also be assigned as homework. The correct answers will be posted and students will vote in a poll on Google Forms to determine which problems they would like more help on. The problems with the most votes will be worked out by the teacher in front of the class. Students could also volunteer to work out or explain those problems. After that, it will be an easy leap to working with graphs that have numbers. Students will complete a worksheet to practice finding limits on graphs with numbers. (This worksheet could replace the prior Know Your Limits worksheet if necessary.)

The next step is to remove the graphs and keep the numbers. Students will watch the Quicksilver video again (without the slow motion part). They will be asked to find the limit as Quicksilver approaches the first police officer. Naturally, this is impossible and students will complain and/or ask to watch the slow motion version of the scene. They will get to watch the slow motion version and they will be asked the same question. What is the limit as he approaches the first police officer? The real goal of this activity is not to find the limit. The big idea is that slowing time down makes it a lot easier to see what is going on. The same is true for functions – looking at an entire graph is a lot of information all at once. If we can slow time down, it will be easier to see what is happening to our function as it approaches a given input. From that introductory discussion, students will do a Think/Pair/Share to discuss what "slowing down time" means in terms of a function and how we could possibly slow down time on a graph, table, and in an equation. This is a major concept of the unit so it is important that students have time to think about and discuss the answers to those questions. After the discussion, students will explore limits numerically using the table on their graphing calculators. In pairs, they will complete the numerical limits worksheet in Appendix 2.

At this point in the unit, students should be familiar with the big idea of a limit, closeness, and infinitesimals. Students will revisit the debate on infinitesimals. They will sort themselves into the same three groups as the first debate and the format of the debate will remain the same. The arguments and evidence for each position may be deeper now, however, since students will incorporate what they have learned about limits. Having this debate before the quiz will help students make connections between all of the information they have learned in this unit. In turn, that will help them understand and apply these concepts better.

After the debate, students will take an individual quiz that focuses not only on the big ideas of the unit but also on the mathematical applications. A sample quiz is included in Appendix 2.

Following the quiz, students will learn how to solve limits with algebra rather than graphs or tables. This is essential due to the nature of the IB Math SL and AP Calculus exams. Students are not allowed to use calculators on the exams and, in most cases, it is a lot more efficient to solve limits with algebra than with other means. However, students can be insecure about their algebra abilities so it is beneficial to start this section with a review of how equations are related to graphs with a particular emphasis on discontinuities. Students should understand that holes and vertical asymptotes are both caused by dividing by zero and that jump discontinuities come from piece-wise functions. These concepts can be reviewed through a lecture or by having them read the relevant part of their textbook.

After students remember how functions are related to their graphs, they will go through a limits activity that has them find limits graphically and then comparing those limits to the function's equation. The equations will get progressively more complex and the graphs will be relied on less and less throughout the activity. By the end of the activity, students will be able to find algebraic limits and use the limit laws to more easily calculate more complex limits. After the exploration, students will practice finding limits using algebra, tables, and graphs. They can be assigned problems from the relevant part of their textbook.

Once students are comfortable with limits, they can move on to applications of limits and, one of the biggest relevant applications is continuity. Class will start with a short story about an engineering competition between universities. Each university was tasked with building a bridge across a river and roads that connected to it. The results of the competition will be shown to the students and there will be a class discussion about what makes a good bridge and set of roads. Students will come up with the informal definition of continuity (in terms of roads and bridges) and then they will, in small groups, figure out how to translate that informal definition into a more formal mathematical definition. The informal definition and its translation are below:

- The roads must go to the same place → $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$
- The bridge has to exist → $f(c)$ exists
- The bridge has to connect to the roads → $\lim_{x \to c} f(x) = f(c)$

Students will then practice determining whether functions are continuous by applying the above definition. They can be assigned problems from the relevant section of their textbook.

The last major topic in the unit is the Intermediate Value Theorem. Students will be given two challenges: first, they must start inside the classroom and then get outside the classroom without going through the door. (This should be impossible for them. The goal is for them to realize that they cannot get outside without going through the door.) The second challenge is for them to get from the back of the classroom to the front of the classroom without climbing over any desks. (This should be trivial for them to do.) After a small number of students have attempted/completed the challenges, students will do a Think/Pair/Share around the following questions: What was the difference between the two challenges? Why was the first one impossible but the second one wasn't? How do those challenges relate to continuity?

When students have come up with the informal idea that the second challenge was only possible because there were gaps between the desks (meaning that the desks were not continuous), the Intermediate Value Theorem can be formally introduced via a lecture. The students already have an intuitive sense of what the IVT means so the instruction should focus on the mathematical language of the theorem along with the importance of a function being continuous (and that, if a function is not continuous, then the IVT does not apply). Students can then practice solving mathematical problems with the Intermediate Value Theorem.

After reviewing the unit, students will fill out a self-assessment and then take a test on the concepts in the unit. After the test is graded and returned, students will fill out the second part of the self-assessment.

Assessments

Quiz

Students will be assessed mid-unit with a quiz on the topics that have been learned so far.

Self-Assessment

Students will fill out a self-assessment about their study habits before taking the test. After the test, they will complete the self-assessment. The second half of this assessment focuses on their performance on the exam and how they can be better prepared for future exams.

Test

At the end of the unit, students will take a test on the whole unit.

Appendix 1: Teaching Standards

IB Math SL

Topic 2.2

The graph of a function; its equation $y = f(x)$. Function graphing skills. Investigation of key features of graphs, such as maximum and minimum values, intercepts, horizontal and vertical asymptotes, symmetry, and consideration of domain and range. Use of technology to graph a variety of functions, including ones not specifically mentioned.

Topic 6.1

Informal ideas of limit and convergence. Limit notation.

Relevance and Rationale

This unit focuses on developing an understanding of limits through the ideas of closeness and the infinitesimal. The more formal delta-epislon definition of a limit is alluded to but not directly taught. Students will instead use the more intuitive "closeness" definition of a limit. Students will start with a graphical approach to understanding limits and then to a numerical and algebraic approach. They will then apply their understanding of limits to continuity and then to the intermediate value theorem.

Appendix 2: Instructional Materials

Quicksilver Videos[18]

Slow Motion included: https://www.youtube.com/watch?v=T9GFyZ5LREQ

No slow motion: https://www.youtube.com/watch?v=w0obSpvhg9k

Quicksilver Think/Pair/Share questions:

- How close did Quicksilver get to the knives?
- How close did the bad guys get to winning?
- How close did Quicksilver get to dying?
- How close did the bullets get to Xavier and Magneto?
- Could those things have been even closer?

Board Meeting Activity

In groups, students will write their answers on their white boards. They should include the definition of "close" as well as real-world and mathematical examples of what "close" means. After filling out their whiteboards, students will stand with their groups in a circle and hold their white boards up so that everyone in the class can see everyone else's whiteboards. (Note: This activity works better with large white boards. Alternately, each group can use 3-4 small whiteboards: one for the definition, one for real-world examples, and one for mathematical examples.)

If the class is large, the class can be split into two large groups with two board meetings running simultaneously. The teacher can be in charge of one group and a specially selected student can be in charge of the other group or students can be in charge of moderating discussions for both groups while the teacher circulates between the groups.

Another variation is to combine the board meeting with a goldfish bowl discussion where half of the class (ideally, two people from each group) take part in the board meeting and the other half of the class stands in a circle around the outside of the board meeting and watches the discussion. Students' roles can be swapped between inner and outer circles midway through the discussion and the discussion should be able to continue since the "goldfish" students were actively listening to the discussion before.

Once the whole class is in the circle, students get 1-3 minutes to silently read all of the boards. After reading the boards, a discussion starts, with the teacher moderating. The suggested questions are below (but teachers are welcome and encouraged to add their own additional questions to deepen/extend the discussion.

- What similarities do you notice?
- What differences do you notice?
- What connections can you make?
- What does "close" mean?

- How close can two numbers/objects be?
- Are the English and math definitions of “close” close to each other?
- (Extension) What happens if two things can’t get any closer?

The key points that teachers should make sure to include in the discussion are listed below. (The students ideally will come up with these in the course of the discussion. The teacher’s job is to ask questions that help the students realize the key points.)

- Essential:
  - There is not a precise definition for “close”
  - Two objects/quantities can always be “closer” together
- Nice to include:
  - If two objects/quantities cannot get closer, then they are the same object/quantity.

Zeno’s Dichotomy Paradox video: https://www.youtube.com/watch?v=EfqVnj-sgcc

Infinitesimal Debate

Students will sort themselves into three groups, depending on their beliefs about the infinitesimal. The groups are: Infinitesimals exist, Infinitesimals do not exist, and It does not matter whether infinitesimals exist or not. In their groups, students will have a set amount of time to come up with arguments, evidence, and examples to support their position. This amount of time can be adjusted to accommodate a variety of schedules. Ten to fifteen minutes is generally enough time to come up with a sufficiently deep argument. After each group has developed their argument, one group is chosen randomly or otherwise to go first. They present their argument and the other groups have a chance to respond. After the other groups respond, the presenting group has time to respond to those responses and conclude their argument. The process is then repeated for each of the other two groups. Then, students can vote on whether they think the infinitesimal exists or not. Voting can be done in several ways: a Google form poll, a Kahoot poll, paper ballots, etc. Students could also physically move around in the classroom to stand with the group that represents their stance on the issue. It is unlikely that there will be a consensus after the debate but this is a conversation that can be revisited throughout the unit and throughout the course.

Gallery Walk: Instructions

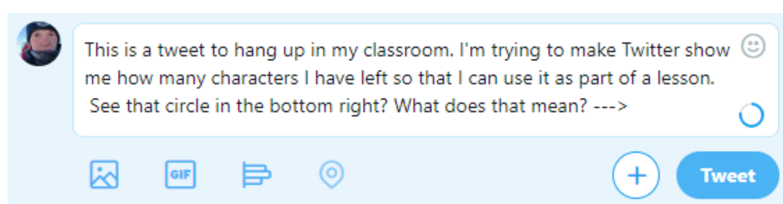
Students will walk around silently looking at the signs. Each student will have some post-it notes that they can write comments on and stick to the signs. They are each expected to write at least 3 comments. Students will be prompted with the following questions:

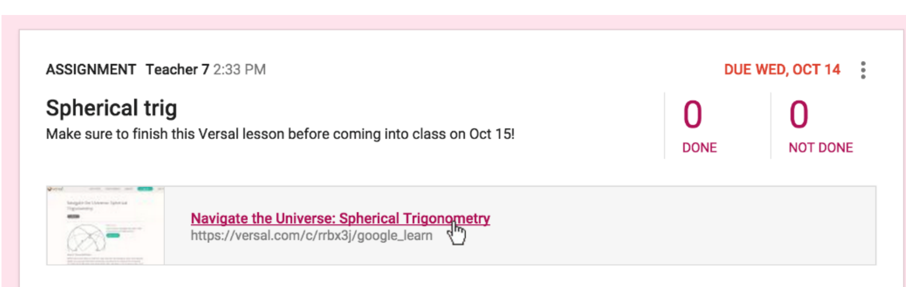
- Do you obey this limit? How much?
- What does this mean in your own life?
- What happens if you disobey this limit?

Students can write their own comments or respond to comments from other students. After students have written their comments and read the signs, they will return to their seats for a class discussion (similar to the board meeting).

Gallery Walk: Signs

Note: Signs should be printed on full sheets of paper (one sign per page)





Limits Exploration

Website: http://jwilson.coe.uga.edu/EMAT6680/Horst/limit/limit.html

Exploration 1: John is craving Taco Bell. John decides to walk to Taco Bell, but decides to travel is a very logical way. John will begin by traveling only half way from his house to Taco Bell in his first hour of walking. Within the second hour, John will travel half the remaining distance from his house to Taco Bell, or one quarter of the distance between his house and Taco Bell. During the third hour, John will travel one-half the remaining distance, or said another way, one eighth of the distance from his house to Taco Bell. Will John ever get to Taco Bell? If so, how long will it take him? If not, how close will he get?

Exploration 2: The Fibonacci sequence is the set of numbers 1, 1, 2, 3, 5, 8, 13, 21, 34, ... where the next number in the sequence is the sum of the previous two (i.e. 21+34=55, so 55 would be the next number in the sequence). Look at the ratios of pairs of consecutive numbers, for example:

$$\frac{1}{1}=1 \quad \frac{2}{1}=2 \quad \frac{3}{2}=1.5 \quad \frac{5}{3}=1.\overline{6}$$

Compute some additional ratios. What happens in the "long run"? Does the sequence of ratios tend to approach a specific value?

Exploration 3: Consider the function below. Is zero in the domain of the function? What happens as this function as *x* gets closer and closer to zero? In other words, is there a limit, or "long run" value that this function approaches as *x* gets close to zero? Begin by exploring this question with your calculator or mathematics software. Compare the graph and table values when you zoom in very close.

$$\left(1+\frac{1}{x}\right)^{x}$$

Know Your Limits Worksheet[19]

A

Calculus Mr. Stadler

Refer to the graph below in order to answer the following questions. If a limit doesn't exist explain why.

1. $\lim_{x \to \infty} g(x) =$

2. $\lim_{x \to -\infty} g(x) =$

3. $\lim_{x \to a^+} g(x) =$

4. $\lim_{x \to a^-} g(x) =$

5. $\lim_{x \to a} g(x) =$

6. $\lim_{x \to 0} g(x) =$

7. $\lim_{x \to b^+} g(x) =$

8. $\lim_{x \to b^-} g(x) =$

9. $\lim_{x \to b} g(x) =$

10. $\lim_{x \to c} g(x) =$

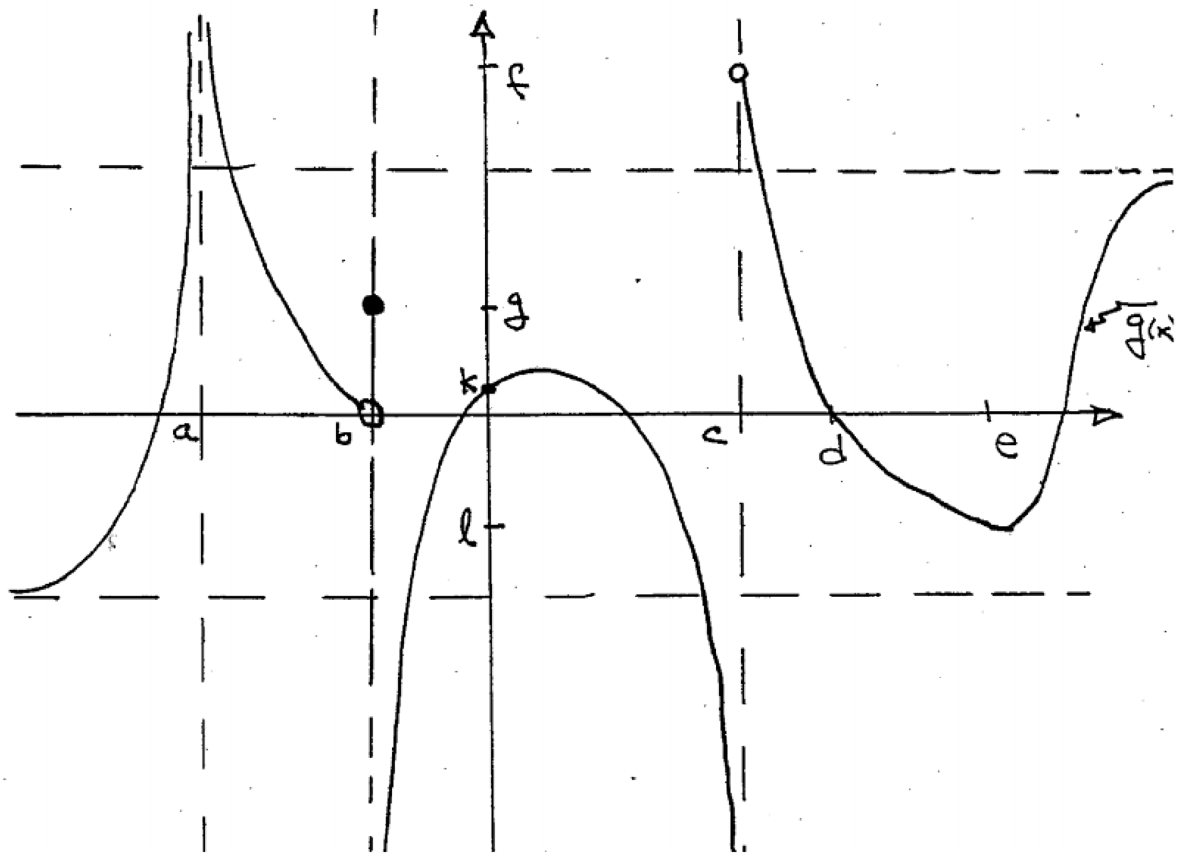
11. $\lim_{x \to d} g(x) =$

12. $\lim_{x \to e} g(x) =$

13. g(e) =

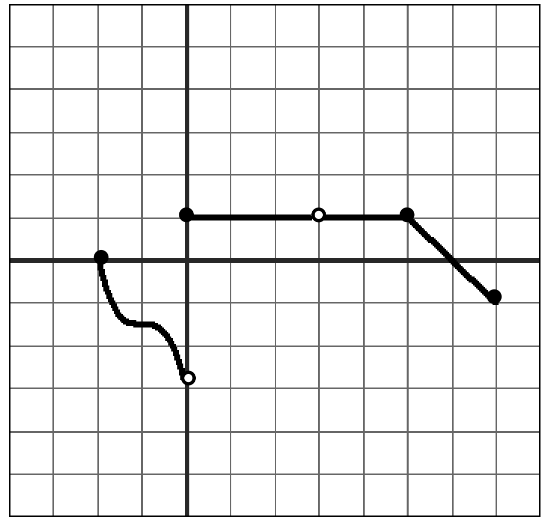
14. g(0) =

15. g(b) =



Graphs with Numbers worksheet[20]

Based on the graph evaluate the following.



1. $\lim_{x\to 0^-} f(x) =$ _____

2. $\lim_{x\to 0^+} f(x) =$ _____

3. $\lim_{x\to 0} f(x) =$ _____

4. $\lim_{x\to 1^-} f(x) =$ _____

5. $\lim_{x\to 1^+} f(x) =$ _____

6. $\lim_{x\to 1} f(x) =$ _____

7. $\lim_{x\to 5} f(x) =$ _____

8. f(1) = _____

9. f(0) = _____

10. f(−2) = _____

11. $\lim_{x\to 6^-} f(x) =$ _____

12. $\lim_{x\to 6^+} f(x) =$ _____

13. $\lim_{x\to 6} f(x) =$ _____

14. f(6) = _____

15. $\lim_{x\to 3} f(x) =$ _____

16. f(3) = _____

17. $\lim_{x\to -1} f(x) \approx$ _____

18. f(−1) ≈ _____

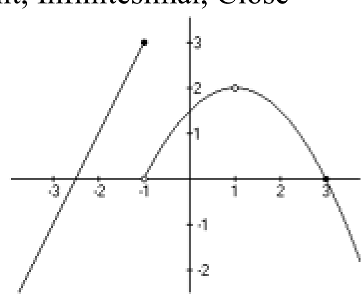
19. <u>True or False</u>: $\lim_{x\to c} f(x)$ exists at every c on (1,3)

20. <u>True or False</u>: $\lim_{x\to c} f(x)$ exists at every c on (−2,1)

Numerical Limits Exploration: http://ntweb.deltastate.edu/vp_academic/cwingard/St.%20Louis--Limits.pdf

Limits Quiz

1. What does limit mean in mathematics? Give a definition in your own words.
2. What is your personal belief about infinitesimals: do they exist? Why or why not?
3. Name two different reasons that a limit would not exist.
4. Which of the words does not belong? Justify your answer. Limit, Infinitesimal, Close
5. Find the following:
   a. $\lim_{x\to -1} f(x)$
   b. $\lim_{x\to 1} f(x)$
   c. $f(-1)$
   d. $\lim_{x\to -1^-} f(x)$



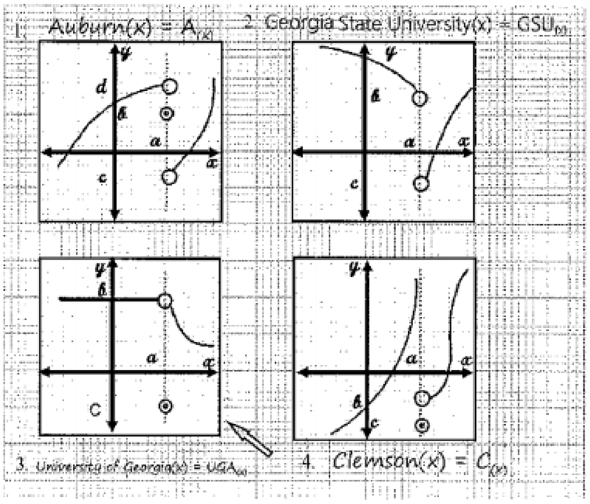
6. Draw one function that has all of the following characteristics:
   a. $\lim_{x\to -2} f(x) = 3$
   b. $f(-2) = 1$
   c. $\lim_{x\to 1^-} f(x) = \infty$
7. a. Explain how you would find the limit of $\left(1+\frac{1}{x}\right)^x$ as x approaches 0.

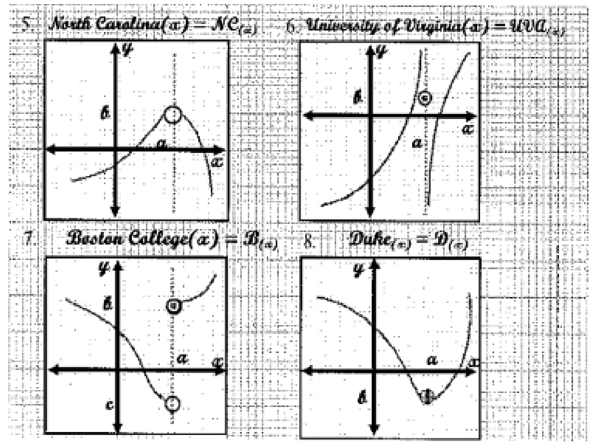
   b. Find the limit of $\left(1+\frac{1}{x}\right)^x$ using the method you described in part a.

Limits: Algebraic Exploration

1. Given the function $y = 5x$:
   a. Graph the function.
   b. What is the limit of the function as x approaches 1?
   c. How does the answer from b relate to the original function?
2. Given the function $y = 5x + 2$:
   a. Graph the function.
   b. What is the limit of the function as x approaches 1?
   c. How does the answer from b relate to the original function?
   d. How does the answer from 2b relate to the answer from 1b? Why do you think that relationship exists?
3. Use what you learned in #2 to find $\lim_{x \to 4} 2x - 1$.
4. Write a general rule for finding the limit of any polynomial at any given x value.
5. Given the function $y = \frac{5}{x}$:
   a. Find the discontinuities. State their types and locations.
   b. Graph the function.
   c. Find the limit as x approaches 0.
   d. How does your answer from part c relate to your answer from part a?
6. Given the function $y = \frac{5}{x^2}$:
   a. Find the discontinuities. State their types and locations.
   b. Graph the function.
   c. Find the limit as x approaches 0.
   d. How does your answer from part c relate to your answer from part a?
   e. How does your answer from 6c relate to your answer from 5c? Why does that relationship exist?
7. How could you find a limit for functions with a vertical asymptote?
8. Given the function $y = \frac{x^2-1}{x+1}$:
   a. Find the discontinuities. State their types and locations.
   b. Graph the function.
   c. Find the limit as x approaches 0.
   d. How does your answer from part c relate to your answer from part a?
9. How could you find a limit for functions with holes?
10. Challenge: Find $\lim_{x \to 4} \frac{1}{\sqrt{x}-2}$

Roads and Bridges Results[21]





Self-Assessment

----------------------------*Before taking the exam*---

1. Approximately how much time did you spend preparing for this exam?

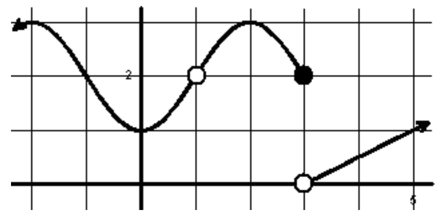
2. What percentage of your test-preparation was spent in each of these activities?
    a. Reading textbook section(s) for the first time
    b. Rereading textbook section(s)
    c. Reviewing homework
    d. Solving problems for practice (on the study guide or otherwise)
    e. Reviewing your own notes
    f. Reviewing materials from course website
    g. Other (Please specify) ________________________________

------------------------------*After the exam is returned* ---

3. Now that you have looked over your graded exam, estimate the percentage of points you lost due to each of the following.
    a. Trouble with (prior knowledge/other concept)
    b. Algebra or arithmetic errors
    c. Lack of understanding of the concept
    d. Not knowing how to approach the problem
    e. Careless mistakes
    f. Other (Please specify) _________________________________
4. Based on your responses to the questions above, name at least <u>three</u> things you plan to do differently in preparing for the next exam.
5. What can we do to help support your learning and your preparation for the next exam?

Test

1. For the graph of *f,* which of the given statements is true?



(A) $\lim_{x \to 3} f(x) = 2$ (B) $\lim_{x \to 2} f(x) = 1$

(C) $\lim_{x \to 1} f(x) = f(3)$ (D) $\lim_{x \to 3} f(x) = 0$

(E) $\lim_{x \to 1} f(x)$ does not exist

2. If $\lim_{x \to 3} h(x) = 7$, which of the following *must* be true?

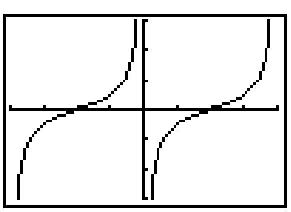
I. $\lim_{x \to 3+} h(x) = 7$ II. $h(3) = 7$ III. $h(x)$ is continuous at $x = 3$.

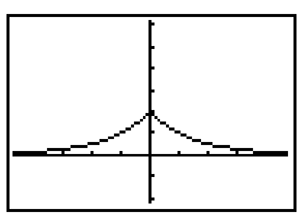
3. Let $g(x) = \begin{cases} x^2 - 4, & x \neq 3 \\ 2, & x = 3 \end{cases}$ Which of the following are true?

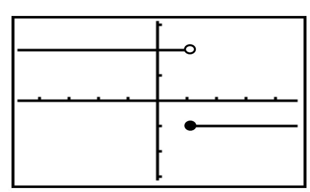
I. g(3) exists II. $\lim_{x \to 3} g(x)$ exists III. g(x) is continuous at x = 3

4. According to the table below $\lim_{x \to 3} f(x)$ appears to be __________.

x	2.5	2.8	2.9	2.99	2.999	2.9999	3.001	3.01	3.1	3.2	3.5
f(x)	0.4	0.357	0.345	0.334	0.333	0.333	0.333	0.332	0.323	0.313	0.286







5.a) $\lim_{x \to 0^-} f(x) =$ _____

b) $\lim_{x \to 0^+} f(x) =$ ______

6.a) $\lim_{x \to 0^-} f(x) =$ ______

b) $\lim_{x \to 0^+} f(x) =$ ______

7.a) $\lim_{x \to 1^-} f(x) =$ ______

b) $\lim_{x \to 1^+} f(x) =$ ______

8. Let $f(x) = 2 + x - x^2$. Use the Intermediate Value Theorem to determine whether there is at least one value for c in [0, 5] such that $f(c) = -4$. Explain your answer. Find the value of c that satisfies the IVT.

9. Let $g(x) = \begin{cases} 3x+1, & x < 0 \\ (x-1)^2, & x > 0 \\ -2, & x = 0 \end{cases}$.

a) $\lim_{x \to 0^+} g(x) =$ b) $\lim_{x \to 0^-} g(x) =$

c) Does $\lim_{x \to 0} g(x)$ exist? Explain.

d) Is g(x) continuous? Justify your answer.

10. $\lim_{x \to 1} (x^3 - 4)$

11. $\lim_{x \to 0} \left(\frac{x-3}{x+1} \right)$

12. $\lim_{x \to -\infty} \left(\frac{5x^3 - 4x^2}{4x^3 - 7} \right)$

13. $lim_{x \to 3} \left(\frac{\sqrt{x+1} - 2}{x - 3} \right)$

14. $\lim_{x \to 2^+} \left(\frac{x^2 - 5x + 6}{x - 2} \right)$

15. $\lim_{x \to 0} \frac{\frac{1}{x+2} - \frac{1}{2}}{x}$

16. $\lim_{x \to -4} \frac{x^2 + 2x - 8}{3x + 12}$

17. $\lim_{x \to a} \frac{x - a}{x^2 - a^2}$ $(a \neq 0)$

18. Find the value of k that makes f(x) continuous .$f(x) = \begin{cases} \frac{4x^2+5x-6}{x+2} & if\ x < 2 \\ 3x + k & if\ x \geq 2 \end{cases}$

Student Resources

Khan Academy - Limits and Continuity: https://www.khanacademy.org/math/calculus-1/cs1-limits-and-continuity

There are a variety of videos, examples, and practice problems for limits and continuity. Students can get immediate feedback about their practice problems as well as getting helpful hints if they get stuck on a problem.

Paul's Online Math Notes – Limits: http://tutorial.math.lamar.edu/Classes/CalcI/LimitsIntro.aspx

A very thorough explanation of limits and continuity. If students want more detailed notes that what they got in class or if they need a different explanation than their textbook gives, this is an excellent site. It explains the big ideas, the informal definitions, the formal definitions, notation, as well as giving examples of each topic.

Teacher Resources

Desmos - Limits and Continuity:
https://teacher.desmos.com/activitybuilder/custom/574de5cdab71b5085a2aad42

This is an extra lesson that can help connect the ideas of continuity and limits for students. Teachers can modify the lesson and give students a class code to access the activity. Teachers can monitor students' progress through the activity.

IB Math SL problem bank: https://kahome.eu/acad/ibprobs.pdf

A booklet full of practice problems for each IB Math SL topic. There are not any specific limit problems but there are a lot of review problems for graphing functions that could be useful for review.

Notes

[1] *East Mecklenburg High*. (2018). Retrieved from USA News and World Report: https://www.usnews.com/education/best-high-schools/north-carolina/districts/charlotte-mecklenburg-schools/east-mecklenburg-high-14567

[2] Alexander, A. (2014). *Infinitesimal.* New York: Scientific American.

[3] Stewart, J. (2008). *Calculus, 7th Edition, AP* Edition.* Belmont: Brooks/Cole.

[4] Alexander, 2014

[5] Alexander, 2014

[6] Alexander, 2014

[7] Alexander, 2014

[8] Alexander, 2014

[9] Mormann, T., & Katz, M. (2013). Infinitesimals as an issue of neo-Kantian philosophy of science. *HOPOS: The Journal of the International Society for the History of Philosophy of Science*, 236-280.

[10] Stix, G. (2018, Summer). Real Time. Scientific American, pp. 5-7.

[11] Davies, P. (2018, Summer). That Mysterious Flow. *Scientific American*, pp. 8-13.

[12] Vonnegut, K. (1969). *Slaughterhouse Five.* New York: Rosetta Books.

[13] Singer, B. (Director). (2014). *X-Men: Days of Future Past* [Motion Picture].

[14] Osborne, T. (2005). Eighteen Minutes. In *Star Trek Voyager: Distant Shores.* New York: Pocket Books.

[15] *Blink of an Eye*. (2018). Retrieved from Memory Alpha: http://memory-alpha.wikia.com/wiki/Blink_of_an_Eye_(episode)

[16] Beaumont, G. (2000). Blink of an Eye. *Star Trek: Voyager.*

[17] Beaumont, 2000

[18] Singer, 2014

[19] Stadler, 2014

[20] Hatboro-Horsham School District, n.d.

[21] Stadler, 2014

Bibliography

(n.d.). Retrieved from Hatboro-Horsham School District: http://www.hatboro-horsham.org/cms/lib2/PA01000027/Centricity/Domain/338/Limit%20Worksheet%20Packet.doc

Alexander, A. (2014). *Infinitesimal.* New York: Scientific American.

Beaumont, G. (2000). Blink of an Eye. *Star Trek: Voyager*.

Blink of an Eye. (2018). Retrieved from Memory Alpha: http://memory-alpha.wikia.com/wiki/Blink_of_an_Eye_(episode)

Davies, P. (2018, Summer). That Mysterious Flow. *Scientific American*, pp. 8-13.

East Mecklenburg High. (2018). Retrieved from USA News and World Report: https://www.usnews.com/education/best-high-schools/north-carolina/districts/charlotte-mecklenburg-schools/east-mecklenburg-high-14567

Mormann, T., & Katz, M. (2013). Infinitesimals as an issue of neo-Kantian philosophy of science. *HOPOS: The Journal of the International Society for the History of Philosophy of Science*, 236-280.

Osborne, T. (2005). Eighteen Minutes. In *Star Trek Voyager: Distant Shores.* New York: Pocket Books.

Singer, B. (Director). (2014). *X-Men: Days of Future Past* [Motion Picture].

Stadler, S. (2014). Know Your Limits. Georgia, USA.

Stewart, J. (2008). *Calculus, 7th Edition, AP* Edition.* Belmont: Brooks/Cole.

Stix, G. (2018, Summer). Real Time. *Scientific American*, pp. 5-7.

Vonnegut, K. (1969). *Slaughterhouse Five.* New York: Rosetta Books.