Mental Math: The Case of the Unnecessary Calculator

Joanne Sato Rowe, 2015 CTI Fellow
Northwest School of the Arts

This curriculum unit is recommended for:
Middle and High School / Math I or II / 8th thru 10th Grade

Keywords: polynomial, perfect square, perfect square trinomial, difference of squares, percent, deduction, fraction, exponent, rational exponent, exponent rules, linear binomials, factoring, structure of an expression

Teaching Standards: See Appendix 1 for teaching standards addressed in this unit.

Synopsis: The purpose of this unit is to provide mental math strategies for students to practice "doing the math in their heads", to develop their numeric sense, and to persevere in problem solving. Although graphing calculators are useful 21st century tools to help students develop graphical understanding of equation solving, they are often used for calculations that replace the mental practice needed to maintain knowledge of how to add, subtract, multiply, and divide integers, fractions, and using exponents. In this unit students train to become detectives by practicing investigative skills and looking at specific cases. Students will work together to do a "Chalk Talk" to identify what they do and do not know about fractions and exponents and use mental math to critique responses. Through "Math Deductions" they will practice using numeric clues to deduce the rules used to find another number and describe in writing what they find. The Cases of the Perfect Square, The Predictable Matrix, and The Deliberate Discount are presented to challenge students to create and test hypotheses by providing mental math multiplication and addition operations with integers that can be extended to polynomials, using percentages and fractions to analyze sales discounts.

I plan to teach this unit during the coming year to **142** students in **Common Core Mathematics II**.

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Background

I currently teach Math II at Northwest School of the Arts (NWSA) which is the only visual and performing arts magnet in the Charlotte Mecklenburg School system. Middle school and high school students must successfully audition or provide a portfolio in order to be considered for the magnet lottery to attend our school. We are a Magnet School of Distinction and our arts program is nationally recognized.

NWSA has a diverse student body and the high school classes that I teach are generally 25% male and 75% female. About half my students are African-American, 40% are white, and 10% are Hispanic, Asian, and mixed races. Classes are set up on an A day/B day schedule in order for students to attend their arts major classes throughout the school year. My honor students are primarily freshman and class sizes vary from 32 to 37 students. My standard students are primarily sophomores and class sizes vary from 20 to 33 students with one class including a co-teacher. Since Math I is an End-of-Course tested subject, students who are struggling in math are often double blocked their freshman year so many of my sophomores had math every day last year. For these students the biggest struggle is to balance a class load of eight classes and have math every other day. Unlike years past, my freshmen were also double blocked last year in middle school due to implementing the Reach Program where the Reach teacher had every 8th grade student every day and the 90 minute class period was split between instruction and going to a computer lab with a Reach associate that monitored their progress.

Our students have strong language arts scores on End of Course assessments and score well on the Math I End of Course assessment but perform lower than the ACT math benchmark. One aspect of their math performance that I would like to focus on is developing their mental math capabilities. This also becomes more crucial since the SAT and the PSAT tests have been updated to include a calculator inactive section with constructed response questions as well as multiple choice questions.

I survey my students at the beginning of the school year and I find that many students do not feel confident in their understanding of math and often fractions will top the list of specific math topics. During my tutoring sessions I have found that standard students feel insecure without a calculator and some have not adequately mastered the multiplication table. This also includes being able to recall the perfect squares of integers up to 12. I have also observed that students don’t begin by asking what a reasonable answer is to a problem and then when using the calculator, students cannot tell whether or not the answer they get on the calculator makes sense. If they have made a mistake entering in the numbers or expressions due to not adequately understanding the order of operations and the need for parenthesis when doing operations with negative numbers, they don't know how to spot the mistake.

Students also do not recognize the opportunities when the answer can be obtained by inspection. An example of solving by inspection, is when determining the discriminant to find the number of real solutions to a quadratic equation. The discriminant, $b^2 - 4ac,$ is an element of the quadratic formula under the radical sign, If b is less than any two of the factors, $4, a, and\ c$, then the answer should be self-evident that the result will be negative and that there are no real solutions since you cannot take the square root of a negative number.

Another common mistake is that when *b* is negative, students forget to put parentheses around the negative number before they square it which results in a negative number on their calculator versus the positive number. Students also want to plug everything at one time into their calculator and using the same example of calculating the discriminant, they only see the final number and are not aware that their answer is incorrect nor where they went wrong in the process.

Content Objectives

The objective of this curriculum unit is to actively engage students in mental math exercises that practice areas of computation that can reinforce their math skills without the use of a calculator. What would be the appropriate timing for this practice and can we associate the practice with our common core state objectives? The current mandate is to end class with an exit ticket to evaluate student understanding and acquisition of the skills needed to solve problems based on the math lesson of the day. Since we have calculators in the classroom students can use, only eight to ten students bring their own calculator to class which requires classroom procedures for students to pick up a calculator at the beginning of class and return their calculator at the end of class after answering the exit ticket. Based on the level of difficulty of the exit ticket, I find that I can have five or six minutes at the end of class to practice one of the mental math strategies.

I also practice different strategies when using PSAT, SAT, and ACT questions as a "warm up" or "do now" to introduce lessons and to provide practice prior to students taking the preliminary tests in the fall. As Marilyn Burns put it, a multiplication problem like 148 × 21 isn't appropriate for mental math if the goal is to find the exact answer but presenting as an estimation problem is a good challenge[1] and many times problems in those types of standardized tests can be answered using estimation and common sense. The summary of these strategies can be found in Appendix 2.

Through the use of puzzles like Mathematical Deductions found in puzzle magazines like Dell Math and Logic Problems[2], students can practice looking for structure where combinations of different mathematical operations such as addition, subtraction, multiplication, and division are used which follows directly with CCSS.HSA.APR.A.1 to understand that polynomials form a system that is closed under the operations of addition, subtraction, and multiplication. The primary mathematical practice is CCSS.MP.7, look for and make use of structure. Putting it in puzzle form helps students to want to make sense of the problems and try to persevere in solving them, CCSS.MP.1.

The next activity is to familiarize students with the concept of perfect squares beyond the single digits that they have already learned. In one case we will explore the squaring of two-digit numbers that end in five and then move into the general case of any two-digit numbers that could be represented as a linear binomial, $10a + b$, where a is the tens digit and b is the ones digit. In both cases there are some mental math strategies that will help students determine them quickly but in addition to applying these methods, I also want students to explore why these strategies work and to see polynomial identities at work to describe numerical relationships (CCSS.HSA.SSE.A.2). Similar problems were presented as a means to help practice the difference of squares but students were hesitant to use new methods when their original methods of multiplying numbers could be used, sometimes presenting materials as a "new trick" helps get beyond that former mindset.

Another problem presented by Harold Reiter, UNCC professor[3], during our seminar to engage his "students" was presented in which there was a "trick" in which he asked us to maximize the sum of the products of a 3 x 3 matrix using the numbers 2, 4, 5, 6, 8, and 9. As part of the presentation of the problem, he circulated around and was able to accurately predict the sum of the products we had given the random order that we placed them in. The combination of multiplication and addition provides practice using some of the mental math addition and multiplication strategies such as breaking numbers into multiples of 10 and single digits and adding from left to right starting with the multiples of 10 first. With multiplication, rounding numbers to the nearest multiple of 10 and then compensating for the rounding. It also lends itself to using the structure of the expressions as a way to rewrite it (CCSS.HSA.SSE.A.2) using the area model since the sum of the three numbers on each side represents the total length on each side of the rectangle being formed and the total of the sums equal the total area of the rectangle.

One of the objectives in Math II is to define the appropriate quantities for the purpose of descriptive modeling, (CCSS.HSA.N.Q.2). One of the most practical uses of rational numbers is associating them with percentages used to calculate restaurant tips and store discounts. Using a real world example of discounts on discounts, we will review fractions, percentages, and discounts and mental math strategies using 10% as a starting point to calculate them quickly to find what we are saving and what we are paying. Connecting the math to money is a practical way for students to make connections to some of the mental calculations that they may already be doing handling money.

Teaching Strategies

Kien H. Lim has written several articles on “Provoking Intellectual Need” and arousing curiosity to make connections[4]. Our seminar leader, Harold Reiter, has also been a great proponent of providing a mystery to engage students to get them motivated to seek patterns and make connections. Growing up I watched a lot of old Perry Mason reruns and his cases always involved a mystery. So in this unit I propose three mysteries, the “Case of the Perfect Squares”, the “Case of the Predictable Matrix”, and the “Case of the Deliberate Discount”.

Problem-Based Tasks

The first two cases involving the perfect squares and a math problem both involve students to observe and look for patterns and relationships. This is one case that I can let students choose their own two digit integers ending in 5 and have them square their numbers manually and see if they can spot the pattern that will make this a mental math problem versus a paper and pencil exercise. Another option, which I also used was to ask students to call a two digit number ending in 5 and tell them automatically what the number is, similar to a magician producing objects out of thin air. The magic act hooks them and their curiosity to figure out the “trick” keeps them interested. It should be noted that at this point I have already shown them the general form for the perfect square trinomial, $(a+b)^2 = a^2 + 2ab + b^2$, but it will be interesting to see if students can explain why the trick work. One of the principles we are trying to encourage with these lessons is perseverance in problem solving and this one is a good example.

There is also a general rule to find the perfect square of any two digit number. Once hooked with the numbers ending in 5, I plan to give the instructions for the short cut that was shared by the Math Dude on his webpage[5] which involves determining the “distance” the number is from the closest multiple of ten. Students can practice and investigate to see if they agree or disagree with the strategy. For students agreeing with the process, they need to then see if they can figure out why it works. To assess students for the special cases there would be a calculator inactive response portion of their unit test. Additional problems would need to be included in future tests to ensure students practice to keep up their skills or can also be used as exit tickets on test days after the calculators have been returned.

For the “Case of the Predictable Matrix”, the problem based approach to find the maximum sum of the products would be used. Students can work in pairs to come up with their possible matrices and to check each other’s calculations using one of the mental math procedures. Students can then Pair and Share their methods and discuss which one works the best for the given situation. It will be interesting to see if any of the students can make the connections to the area model to visually see how to maximize the solution before guided questions are given to help reach that conclusion.

I have also included an exercise called “Math Deductions” which is also a problem based activity. Students are given two examples of two sets of numbers. Their job is to find the relationship in the form of math operations that will connect the three numbers to the number in the box. As stated before, this is good practice in breaking down a number or trying to come up with its structure based on the information given. Another key element of this exercise is to practice writing down what is occurring mathematically. Students are used to treating math as learning algorithms and applying them but are seldom asked to explain in writing what they are trying to accomplish using mathematical terms.

Visualize Thinking

As a pre-assessment for my lesson on rational exponents and to help drive classroom discussion, I would like students to take part in a Chalk Talk. Emily Sliman talks about visualizing student thinking[6] by providing them with a topic, markers and poster paper, and the caveat that all communication must be handwritten on the paper with no talking. Students need to brainstorm and explain ideas but they can’t talk. Examples and pictures can also be used to express concepts. I see this as a fun pre-assessment to find out what students know and understand in regards to fractions and exponents. The mental math comes with those students who chose to explain concepts by giving examples and for those students who read and critique what other students have written before them. Sliman states that it is good to include several stations with specific questions. The discussion which follows on paper and verbally afterwards should help students with rules involving math operations with fractions and exponents. It also helps me identify what misconceptions that students have with both of those concepts.

Inquiry-based Learning

With the “Case of the Deliberate Discount”, one of James R. Olson’s key strategies will be fulfilled which is usefulness[7]. Of all the problems, this one can be taken out of the classroom and observed at just about any large department store that sets up different sections for sales and discounts. Media Clips are featured in the Mathematics Teacher magazine and in the November 2014 issue it included a post from another magazine that showed a Kohl’s sign that said clearance items reduced 60% would be reduced an additional 25% for a total of 85% off.[8] Ads like these are in department stores all of the time and students should be able to decide if the discount they are receiving is correct or not. Although I originally thought of presenting this as a problem based task, I also thought it could fall under inquiry-based learning.

Strategies involving inquiry-based learning consist of five parts[9]. The first is questioning and in this case are there other examples of false advertising that involves math. The second part is planning and predicting and students can predict whether this was a fluke or does it happen in other situations. The third part is investigating and this is where students can either check newspaper ads, online sales, and actual stores. The fourth part is recording and reporting their findings. I provided this activity as an extra credit project on the eve of Black Friday with a template they can fill out with their findings. The last part is reflection where they can form conclusions based on their findings. An extension of this would be to also include visits to restaurants and to check the bill and tip given.

The culmination of this portion would be a group activity with different stations based on the students' findings, and for students to be given an allowance and to spend it at the different stations and determine if they have the correct change at the end of the exercise.

Classroom Activities

The Case of the Predictable Matrix

Prior to this activity, some of the mental math addition strategies have been reviewed. See Appendix 2. With this problem, students are given six specific numbers (2, 4, 5, 6, 8, 9) and are asked to draw a 3 x 3 matrix and using each number only once, place one of the numbers either above a column or to the left of a row. Once numbers are placed, their product goes inside the box using the Box method. For example, see below:

	6	8	9
2	12	16	18
4	24	32	36
5	30	40	45

Once the products have been calculated mentally, add up all of the products. The goal is to find the largest possible sum of the products.

After students have calculated their sums, take a look at their numbers and predict what their answer should be. Teachers can do this by realizing that what is actually occurring is that a rectangle is being formed, the length is the sum of the numbers across the top and the width is the sum of the numbers down the side. So in this example, without having to add the individual boxes up, the teacher can recognize this is an 11 x 23 box. Using the mental math strategy for multiplication by 11, the first digit of the number being multiplied is the first digit, the second digit is the sum of the digits, and the last digit is the last digit of the number being multiplied. Thus by mental math the product of 11 and 23 is 253 which should also be the number calculated by the student using the individual parts.

Teachers can always use a calculator or a cheat sheet to check answers but it is good practice for the teacher to practice mental math as well. Part of the goal of this exercise is to see if students can make the visual connection to the area model on their own. The area model is one way to visualize the structure of the numbers as they are being multiplied and Khan Academy[10] has several videos with examples that can be used if students are still struggling with the concept. Using a uniform matrix is intentional to set the focus on the number. Once the students understand that visual aspect of the area model, then the problem transitions from producing the biggest number to forming the biggest area. The intent here is to extend the area model to work with the multiplication of polynomials. Both the box method and the place value method are also shown when multiplying binomials and trinomials.

Given the graphical aspect of maximizing a rectangle, the correct answer will be a square model. The numbers 2, 4, 5, 6, 8, and 9 add up to 34 and since half of 34 is 17, the correct answer will be 289 and the sides should be grouped (4, 5, 8) and (2, 6, 9) with both sets adding up to 17.

The Case of the Perfect Squares

Students do need to be reminded what constitutes a perfect square, both numerically as well as algebraically. Most Math I students have been taught binomial perfect squares but they need a reminder by the time they reach Math II. As a reminder the rule is:

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$

There are two possible ways to introduce this activity. The first is to have students take several of the two digit numbers that end in five and square them without using a calculator and see if they can spot a pattern. The other way which I have used due to time considerations is having students volunteer the two digit numbers and for me to write them immediately on the board. Although the first approach touches on the investigative means of personally finding the rule, the second approach is far more engaging and can take on the appearance of a stage act. The honors class will usually come up with at least one person to spot the trick after three numbers but a standard class may need some prompting.

Given a number like 65, the square of 65 will have 25 in the last two digits of the number, and the first two digits will be the product of the 10’s digit, which is 6 multiplied by the number that comes next which is 7. Their product is 42 so the square of 65 is 4225.

The trick is that if a represents the digit in the 10's place, then the number can be expressed as the linear binomial, $10a + b.$ The last two digits of the perfect square will always be 25 since the last digit of the number, *b,* is equal to 5. Since the tens digit will always be a single digit followed by a zero, you square the digit and follow it by 00. But note that there is always the middle term and that includes twice the product of 10*a* and *b* which will always be equal to 100*a.* Factor $(100a^2 + 100a) = 100a(a + 1)$ or 100 times the product of the tens digit and the number following it. This can lead to looking at the more general case of any given two digit number. Students want to use the rule used with 5 for any given two digit number but remember that the trick only works with 5 because the middle term has a product of 10 × 10 = 100.

For the general case, the rule is to turn the number being multiplied into product of their conjugates plus the square of the number being added or subtracted. For example, 33^2= (33 – 3) (33 + 3) + 3^2 = (30×36) + 9 = 1080 + 9 = 1089. Students can investigate to see if this is always the case and to determine why it works. Note that you are re-writing the given number into the factored form of the difference of squares and by adding back the last term squared you are making it equate to the original number. The only difference is that the math is easier if you turn one of the numbers into a multiple of 10 before you multiply.

Fractions and Exponent Chalk Talk

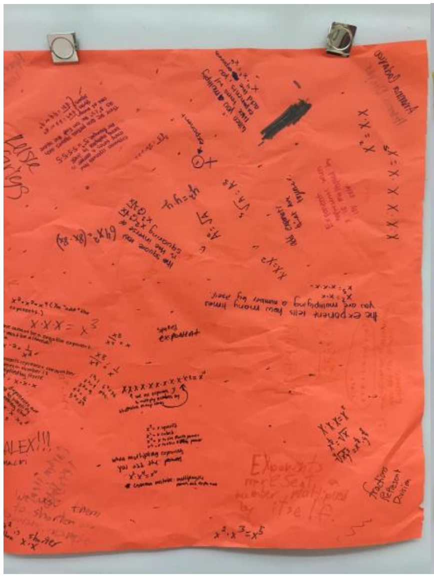
Materials Needed: Chart paper for each group
(I used 2 different colors to distinguish fractions from exponents)
Box of markers for each group
Timer (or online clock)
Instruction Sheet with guided questions for each group (Appendix 3)

Group desks so it is easy for everyone to be able to write on the chart paper. I allowed students to choose their own groups. If you don't wish to do this, you can have a different color marker for each group and hand them to students as they walk in the classroom.

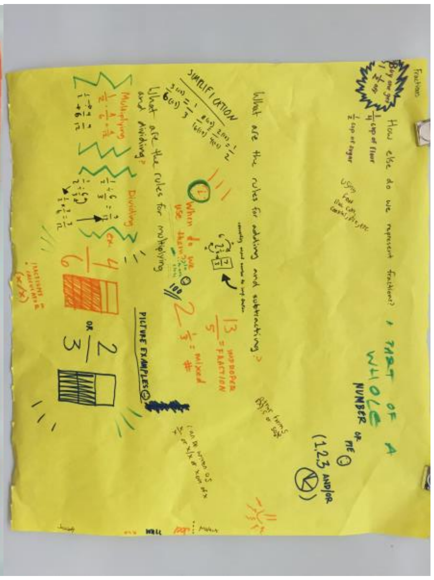
If this is the first time students have done this activity, I set the timer after everyone looks like they understand the objective of the activity and talking has ceased. At five minutes I ask students to stop and then examine or question on paper or critique or add something to what someone else has already written and I give them another five minutes.

At this point I ask groups with different topics to switch with another group and we repeat the process. Since the chart paper already has notes on it, the first five minutes are for reading and critiquing and the last five minutes for additional notes or examples.

Students should be using their mental math skills to check on any examples.



Exponents



Fractions

Math Deductions

The timing for this activity is flexible. I used it in conjunction with the Case of the Deliberate Discount so that students had something engaging to do prior to the Thanksgiving Holiday that did not involve their calculator

The worksheet and the answer key are in Appendix 4 and are self-explanatory. Students examine the two examples given and try to figure out the rule to produce the correct number in the last box and then they are to write the rule down in the space below with as little as possible guidance to see how they would describe their rules. I do tell them that the math operations are addition, subtraction, multiplication, and division and that the rule has to include all three of the numbers. The problems do get increasingly harder and the one clue that I do give if students are struggling is that “position” of the numbers is no longer relevant as it was in the first three problems. They need to look for other types of relationships the numbers may have with each other.

Have students present their solutions and read their descriptions. If time permits or if students are having a difficult time with the task, have them pair and share with a peer to see if the other student can help them see a possible solution. Before collecting papers, make sure a few minutes are given so that students who needed help have a chance to provide a written explanation in their own words after they have heard some of the solutions.

Since the goal was pattern recognition, I also gave them the first nine Fibonacci numbers (0, 1, 1, 2, 3, 5, 8, 13, 21) to see if they could spot the pattern and to write down the next five numbers in the sequence using mental math. There is an excellent website I used connecting the Fibonacci numbers with the Golden Ratio and the Golden Rectangle (http://www.homeschoolmath.net/teaching/fibonacci_golden_section.php.) I followed reviewing the website with the YouTube video of a portion of Donald in Mathmagic Land. (https://www.youtube.com/watch?v=U_ZHsk0-eF0)

The Case of the Deliberate Discount

Each student is given a handout which describes the purpose of the case and some of the mental math skills needed to complete this assignment. On the reverse side of the handout is a template students can use to go out and investigate actual sales signs and restaurant bills. These worksheets are in Appendix 5. This activity was done in conjunction with Math Deduction so that students who finished that activity early could work on this activity. I also allowed online sales to be used as well so students with their own technology could get started if time permits.

The key mental math skills used for this activity are finding the percentages of different amounts starting with 10% as our base, multiples of 10%, and multiples of 5%. Since the percentages represent discounts, there are two ways to determine what the final price is, identify the discount and subtract, or find the complement of the percent discount and just multiply. Most students do not make the connection of finding the percent that you do pay with finding what you get off as a discount. Once it is pointed out to them, many recognize the value of this strategy. This is also helpful when we go into our probability unit.

Once students get a chance to master the mental math strategies then as an exit ticket the contents of the actual store sign is presented.

Clearance
85% off
When you take an additional 25% off
Prices already reduced 60% off

The original price of a shirt was $80.

What do you think the new price is based on the sign? ($8 + $4 = $12)
What method did you use to determine that price? (15% is what was paid)
What is the actual price they will charge you at the register? $24
How did you calculate that price? 40% of $80 = $32 - $8 = $24

Original price of another shirt was $36. Answer the same questions above for that item.

What do you think the new price is based on the sign? ($3.60 + $1.80 = $5.40)
What method did you use to determine that price? (15% is what was paid)
What is the actual price they will charge you at the register? $24
How did you calculate that price? 40% of $36 = $14.40 - $3.60 = $10.80

As an extra credit inquiry-based task, students are asked to find actual sales signs, checking to see if they are correct. They can also include restaurant bills and the calculation of the tip. The form is included in Appendix 5. The students can provide a brief written explanation of their finds and conclusions. As a review for this lesson, some of the actual examples will be used for a group activity where stations are going to be set up using some of the actual examples. The assessment for this would be to “give” each student an allowance and check their final amount after checking out all of the stations.

Final Notes

Now that I am entering into my third year teaching Math II, I see a re-occurring theme in first units that I teach, it is all about taking the structure of numbers, breaking them down into addition, multiplication, subtraction, or division, fractional and exponential expressions, and applying those same rules to algebraic expressions. By better understanding the structure of numbers, they can rewrite expressions to help them understand the concepts of factoring expressions to solve quadratic equations, multiplying multiple term polynomials to model problems such as area and volume, and simplifying rational expressions to better understand rate problems.

Appendix 1: Implementing Common Core Standards

This unit incorporates the North Carolina Common Core Standards for Math II. Many of these standards are also included in Math I.

CCSS.Math.Content.HSA.APR.A.1 Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

> When students are confident in performing math operations with numbers, they find it easier to apply those strategies with operations involving polynomials.

CCSS.Math.Content.HSA.SSE.A.1 Interpret expressions that represent a quantity in terms of its context.*

CCSS.Math.Content.HSA.SSE.A.2 Use the structure of an expression to identify ways to rewrite it. *For example, see* $x^4 - y^4$ *as* $(x^2)^2 - (y^2)^2$*, thus recognizing it as a difference of squares that can be factored as* $(x^2 - y^2)(x^2 + y^2)$

> By learning how to rewrite integers as a product of its factors, or as a sum or difference of two numbers, students make connections to rewrite polynomials.

CCSS.Math.Content.HSN.RN.A.2 Rewrite expressions involving radicals and rational exponents using the properties of exponents.

> When applying the exponent rules to rational exponents, students will need to know how to add, subtract, and multiply fractions.

CCSS.Math.Content.HSN.Q.A.1 Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.

CCSS.Math.Content.HSN.Q.A.2 Define appropriate quantities for the purpose of descriptive modeling.

> Part of analyzing a problem is to define and use the appropriate units to describe the problem which is highlighted in the problem with extremely large numbers.

Appendix 2: List of Suggested Mental Math Strategies

See the List of Teacher and Student Resource List for the Sources in Parentheses

Addition	1. When given a group of single digit numbers to add, first group numbers that add up to 10 ("ten" pairs). (Math Dude #1) Ex. 3 + 8 + 7 + 6 + 2 = (3 + 7) + (8 + 2) + 6 = 26 2. Work from left to right (Math Dude #2) Ex. 543 + 285 +161 = (500 + 200 + 100) + (40 + 80 + 60) + (3 + 5 + 1) = 800 + 180 + 9 = 989 3. Compatible numbers for Addition, group pairs of numbers that add up to multiples of 10 (Olson #1) Ex. 34 + 43 + 66 + 7 = (34 +66) + (43 + 7) = 150 4. Compensate by rounding numbers up or down to nearest multiple of 10 and then adding or subtracting back. (Olson #4) Ex. 53 + 48 = 53 + 50 – 2 = 103 – 2 = 101 4.99 + 11.95 = (5 + 12) – (.01 + .05) = 16.94 5. Treat the amounts as money and break down their structure in terms of pennies, nickels, dimes, and quarters. Or by bills as $1, $5, $10, or $20. Ex. 225 + 719 = 9 + quarter + dime + nickel + 4 pennies = 944
Subtraction	1. Equal addition or subtraction of number to the next multiple of 10 (Olson #7) Ex. 84 – 47 = (84 + 3) – (47 + 3) = 87 – 50 = 37 72 – 33 = (72 – 3) – (33 – 3) = 69 – 30 = 39 2. Treat the amounts as money (see Addition above) and break down amounts into each of the denominations or coins and count up similar to giving change back. Ex. 3052 - 580 = 2 dimes takes you to $6, $24 takes you to $30, and 2 quarters and 2 pennies takes you to $30.52. $24 + 2 quarters + 2 dimes + 2 pennies = 2472 3. Subtract from 1,000 starting at the left, subtract each digit from 9 except for the last digit which is subtracted from 10 (10 Tricks Website) Ex. 1000 – 672 = (9-6) (9-7) (10-2) = 3 2 8 = 328 4. Break the number that is being subtracted into sum with a multiple of 10 (Math Dude) Ex. 187 – 48 = 187 – 40 – 8 = 147 – 8 = 139

Multiplication

1. Multiply by 10, add the zero at the end
   Ex. 95 × 10 = 950
2. Multiply by 5, (multiply by 10 first, then take half)
   Ex. 47 × 5 = (47 × 10) ÷ 2 = 470 ÷ 2 = 235
   If the factor is even, divide it by 2 and then multiply by 10
   Ex. 128 × 5 = (128 ÷ 2) × 10 = 64 × 10 = 640
3. Multiply by 9, (multiply by 10, then take one away)
   Ex. 15 × 9 = 15 × 10 – 15 = 150 – 15 = 135
4. Double when 2 is a factor, double again when 4 is a factor, double again if 8 is a factor
   Ex. 32 × 2 = 64
   32 × 4 = 64 × 2 = 128
   32 × 8 = 128 × 2 = 256
5. Both numbers end in zeroes. Multiply the numbers and add the number of zeroes from both numbers to the end.
   Ex. 50 × 80 = (5 × 8) = 40 add 00 at end to make 4000
6. Distribute (break down in terms of powers of 10's) (Math Dude #3)
   Ex. 4 × 56 = (4 × 50) + (4 × 6) = 200 + 24 = 224
7. Re-write numbers using structure of the difference of squares
   Ex. 54 x 66 can be re-written as (60 - 6)(60 + 6) and becomes the difference of squares 3600 – 36 or 3564
8. Take half of one number and double the other number to see if numbers are easier to compute
   Ex. 25 × 60 = (25 × 2) (60 ÷ 2) = 50 × 30 = 1500
9. Square a two digit numbers that ends in 5. Last two digits will be 25, the other two will be the product of the ten's place plus one. (Math Dude)
   Ex. $65 \times 65 = \underline{(6 \times 7)}\ \ \underline{2}\ \underline{5} = 4225$
10. Multiplying two digit numbers by 11, the middle digit will be the sum of the two digits number being multiplied by 11.
    Ex. $35 \text{ x } 11,\ \underline{3}\ \underline{(3+5)}\ \underline{5} = 385$
    If the sum of the two digits in the middle is greater than 9, then the number in the front moves up one and the middle number is the remainder. (10 Tricks Website)
    Ex. $67 \text{ x } 11 = \underline{(6+1)}\ \underline{(3)}\ \underline{7} = 737$
11. Squaring any two digit numbers by finding the distance to closest multiple of 10, add distance to one factor and subtract it from the other factor. Add the square of the distance to product. (Math Dude)
    Ex. $23 \times 23 = (23 - 3)(23 + 3) + (3^2) = (20)(26) + 9 = 520 + 9 = 529$

Division	1. Dividing by 10 is equivalent to taking 10% of a number. Figure percentages starting with 10% by moving the decimal point to the left one place. (Math Dude #4) Ex. 10% of 55 is 5.5 2. Figure out percentage in relation to 10%, 5% is half, 20% is twice, 15% is 10% + 5% Ex. 5% of 60 is half of 10%, half of 6 = 3 20% of 60 is taking 6 twice = 12 15% of 60 is adding 5% + 10% = 6 + 3 = 9 3. Divide by 2, cut in half; divide by 4, cut in half twice; divide by 8, cut in half 3 times Ex. 120 ÷2 = 60 120 ÷ 4 = 30 120 ÷ 8 = 15 4. Divide by a decimal, multiply both numerator and denominator by 10 or 100 to get rid of decimal Ex. 48 ÷ .4 = 480 ÷ 4 = 120 5. To divide by 5, double number and divide by 10 Ex. 36 ÷ 5 = (36 × 2) ÷ 10 = 72 ÷10 = 7.2 6. Division tricks - Here's a quick way to know when a number can be evenly divided by these certain numbers: (10 Tricks website) • 10 if the number ends in 0 • 9 when the digits are added together and the total is evenly divisible by 9 • 8 if the last three digits are evenly divisible by 8 or are 000 • 6 if it is an even number and when the digits are added together the answer is evenly divisible by 3 • 5 if it ends in a 0 or 5 • 4 if it ends in 00 or a two digit number that is evenly divisible by 4 • 3 when the digits are added together and the result is evenly divisible by the number 3 • 2 if it ends in 0, 2, 4, 6, or 8

Appendix 3

Intro to Rational Exponents

Chalk Talk - SILENCE, PLEASE! The only talking happens on paper!

- **This is a free-writing exercise and everyone is expected to participate so make sure everyone is in a position to be writing on the paper**
- **You may use pencils, colored pencils, markers, pens, or highlighter**
- **First 5 minutes free write or draw on the topic using the following guided questions.**
- **Second 5 minutes, consider what the rest of your group has written. Do you agree? Disagree? Need more information? See if someone in your group can write an explanation. See if you can expand on or illustrate what someone else has written.**
- **When the second 5 minutes is up, exchange chart paper with a group that has a different topic and repeat the process.**

FRACTIONS

What do they represent?

How else do we represent fractions?

What are the rules for adding and subtracting?

What are the rules for multiplying and dividing?

SHOW EXAMPLES, DRAW PICTURES, AND EXPLAIN IN WRITING ON THE PAPER

EXPONENTS

What do they represent?

When do we use them?

What are the rules for Exponents?

SHOW EXAMPLES, DRAW PICTURES, AND EXPLAIN IN WRITING ON THE PAPER

Appendix 4 – **Math Deductions** -To practice your "deductive skills", in each row below there is a rule involving math operations using the three numbers surrounding the number in the block. Two examples are given and it is your job to figure out the rule, write it down below, and predict the number in the last block.

1)	7	19	3	5	14	4	8		4
		2			6			5	
2)	2	20	4	6	21	3	7		4
		3			1			3	
3)	1	7	6	4	13	9	2		7
		2			8			5	
4)	14	10	3	9	12	30	12		2
		2			10			7	
5)	1	7	12	24	12	3	2		15
		5			4			5	

Write down the rule:

1) ______________________________
2) ______________________________
3) ______________________________
4) ______________________________
5) ______________________________

To practice your “deductive skills”, in each row below there is a rule involving math operations using the three numbers surrounding the number in the block. Two examples are given and it is your job to figure out the rule, write it down below, and predict the number in the last block.

1)	7	19	3	5	14	4	8	27	4
		2			6			5	
2)	2	20	4	6	21	3	7	40	4
		3			1			3	
3)	1	7	6	4	13	9	2	10	7
		2			8			5	
4)	14	10	3	9	12	30	12	13	2
		2			10			7	
5)	1	7	12	24	12	3	2	5	15
		5			4			5	

Write down the rule:

1) Multiply left and right numbers and then subtract bottom number
2) Add the left and bottom numbers and multiply sum by right number
3) Add bottom and right numbers and then subtract the left number
4) Divide the largest even number by the smallest and add the odd number
5) Subtract the product of the two smaller numbers from the largest

Appendix 5: The Case of the Deliberate Discount

Now you are going to take your deductive skills and work on some sales being advertised for Black Friday, known by shoppers to have the deepest discounts on top selling products.

Sometimes you need to make spur of the moment decisions to purchase items due to limited supply or a small sales ‘window” when the discount is being offered for a limited time. To get prepared, we need to sharpen our mental math skills so that we can decide quickly if what is being offered is a “good” deal, especially if it is discount or sale on top of what is already being discounted.

Typical sales are usually multiples of 10 like 10%, 20%, 30% or up to 70% or 80% off. Which way is easier for you to figure out what the new price is?

Example) A $60 coat is 30% off.

1) **Move the decimal one place to the left to find 1/10th, multiply to find the percent off and then subtract from the original price.**
   Ex) 10% of 60 is 6, multiply by 3 (30%) is 18, subtract 18 from 60 to get $42.

2) **Move the decimal one place to the left to find 1/10th, subtract the percentage off from 100% to find what is left to pay, and then multiply the 1/10th by the percentage left.**
   Ex) 10% of 60 is 6, 30% from 100% is 70%, 6 multiplied by 7 is $42.

3) **If the discount is 50% then just take half of the total amount**

When sales are multiples of 5’s like 15%, 25%, 35%, etc., then find 5% by taking half of 10% and adding it to the multiple of 10. This is useful when finding a 15% tip at a restaurant. Take 10% of the total and figure out half of that amount. Add it to the 10% and now you have 15%.

In special cases where it is 25% off sale, figure out what is half off and then take half of that. Remember ¼ or 25% is half of ½ or 50%. Remember to subtract it from the total unless the sale is 75% off. In that case, half of a half is what you pay.

Objective: Visit either a store or a restaurant and mentally figure out the sales price or tip. Test the Truth in Advertising, if it is not true, figure out what the correct sales price should be.

Store/Restaurant: Original Price or Restaurant Total %Discount or Tip: Total Price before taxes:	Store/Restaurant: Original Price or Restaurant Total %Discount or Tip: Total Price before taxes:
Store/Restaurant: Original Price or Restaurant Total %Discount or Tip: Total Price before taxes:	Store/Restaurant: Original Price or Restaurant Total %Discount or Tip: Total Price before taxes:
Store/Restaurant: Original Price or Restaurant Total %Discount or Tip: Total Price before taxes:	Store/Restaurant: Original Price or Restaurant Total %Discount or Tip: Total Price before taxes:

Resources for Teachers and Students

“5 Tips for Faster Mental Multiplication.” *Quick and Dirty Tips*. Accessed September 26, 2015. http://www.quickanddirtytips.com/education/math/5-tips-for-faster-mental-multiplication.
Mental math strategy for multiplying by numbers ending in 5.

“10 Tricks for Doing Math Quickly in Your Head.” Accessed September 26, 2015. http://education.cu-portland.edu/blog/educator-tips/10-tricks-for-doing-fast-math/.
Additional mental math strategies, see Appendix 2.

“Do Speedy Math in Your Head - Wired How-To Wiki.” Accessed September 26, 2015. http://howto.wired.com/wiki/Do_Speedy_Math_in_Your_Head.
Mental math strategies.

“How to Square Two-Digit Numbers in Your Head.” *Quick and Dirty Tips*. Accessed November 2, 2015. http://www.quickanddirtytips.com/education/math/how-to-square-two-digit-numbers-in-your-head.
Mental math process on how to square two-digit numbers.

“Khan Academy.” *Khan Academy*. Accessed November 30, 2015. http://www.khanacademy.org.
Source for explanations of multiplication using the area model.

Marshall, Jason. 2011. *The Math Dude’s 5 Tips to Mastering Mental Math*. Macmillan Audio.
Tips are included in Appendix 2 which summarizes various mental math strategies.

Materials for Classroom

“Lesson Plan for Fibonacci Numbers, the Golden Section, the Golden Ratio, and Phi.” Accessed November 26, 2015. http://www.homeschoolmath.net/teaching/fibonacci_golden_section.php.
Website with additional information on Fibonacci numbers that includes a graphical representation and its connection to the Golden Ratio and Rectangle.

Reiter, Harold B. “Harold Reiter’s Home Page.” Accessed November 27, 2015. http://math2.uncc.edu/~hbreiter/.
Seminar leader’s website with online links to additional math problems, math problem solving, and math organizations.

TrueDisney. *Donald Duck - Mathmagic Land*, 2012. https://www.youtube.com/watch?v=U_ZHsk0-eF0.

Release date 6/26/59 by Walt Disney Productions, run time is 27 minutes, color. Connects math to Pythagoras, our current music scale, and to the Golden Rectangle. This section starts at 2:00 and ends around 13:00 minutes.

Bibliography for Teachers

Brown, Scott A., and Cheryl L. Avila. "Discounts on Discounts," Mathematics Teacher, 108, no. 4 (November 2014): 250.

Discussion and questions relating to an actual Kohl's advertisement misusing discount percentages.

Burns, Marilyn. "Marilyn Burns: Mental Math." *Instructor* 116, no. 6 (March–April 2007): 51–54.

Use of mental math for estimation.

Lim, Kien H. "Curiosity and Connections." *The Mathematics Teacher* 108, no. 4 (2014): 288–95. doi:10.5951/mathteacher.108.4.0288.

Need for student engagement to aid retaining math concepts by providing problems that appeal to their curiosity.

Lim, Kien H. "Provoking Intellectual Need." *Mathematics Teaching in the Middle School* 15, no. 2 (September 1, 2009): 92–99.

The Necessity Principle for students to learn and apply math concept with focus on prime factorization and finding the least common multiple.

"Mathematical Deduction." *Dell Math and Logic Problems*, November 2015.

Math deduction puzzle based on one of the puzzles in this puzzle magazine.

"Mathematics Standards | Common Core State Standards Initiative." Accessed November 27, 2015. http://www.corestandards.org/Math/.

Hyperlinks in Appendix 1 to this website on the math common core standards.

Northrup, Laura. "At Kohl's, 60% Off, Then 25% Off Means 85% Off." *Consumerist*, August 11, 2013.

———. "Kohl's Employee Explains Math-Impaired Signage." *Consumerist*, August 11, 2013.

Article referenced in the Mathematics Teacher Media Clip about the concern one customer had about the misleading Kohl's sale sign.

Olsen, James R. "Five Keys for Teaching Mental Math." *Mathematics Teacher* 108, no. 7 (March 1, 2015): 543–47.

The five keys are Usefulness, Strategies, Practice, Decision, and Mindset. Strategies include compatible numbers, break apart, and compensation.

Seeley, Cathy L. “Do the Math in Your Head!” The National Council of Teachers of Mathematics News Bulletin, December 2005. http://www.nctm.org/News-and-Calendar/Messages-from-the-President/Archive/Cathy-Seeley/Do-the-Math-in-Your-Head!/.
Notes on some of the mental math computations that are expected in high school.

Sliman, Emily. “Visible Thinking in High School Mathematics.” *Mathematics Teacher* 106, no. 7 (March 1, 2013): 502–7.
Chalk Talk strategy for students to write down what they know and share it with others.

“TWT: Inquiry-Based Learning Strategy.” Accessed November 3, 2015. http://www.ndtwt.org/Blackboard/P2SST2/inqu.htm.
Overview of the process of teaching student through inquiry-based learning strategies.

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[1] Burns, Marilyn. “Marilyn Burns: Mental Math.” *Instructor* 116, no. 6 (March–April 2007): 51–54.

[2] “Mathematical Deduction.” *Dell Math and Logic Problems*, November 2015.
Math deduction puzzle based on one of the puzzles in this puzzle magazine.

[3] Reiter, Harold B. “Harold Reiter’s Home Page.” Accessed November 27, 2015. http://math2.uncc.edu/~hbreiter/.

[4] Lim, Kien H. “Provoking Intellectual Need.” *Mathematics Teaching in the Middle School* 15, no. 2 (September 1, 2009): 92–99.

[5] “5 Tips for Faster Mental Multiplication.” *Quick and Dirty Tips*. Accessed September 26, 2015. http://www.quickanddirtytips.com/education/math/5-tips-for-faster-mental-multiplication.

[6] Sliman, Emily. “Visible Thinking in High School Mathematics.” *Mathematics Teacher* 106, no. 7 (March 1, 2013): 502–7.

[7] Olsen, James R. “Five Keys for Teaching Mental Math.” *Mathematics Teacher* 108, no. 7 (March 1, 2015): 543–47.

[8] Northrup, Laura. “At Kohl’s, 60% Off, Then 25% Off Means 85% Off.” *Consumerist*, August 11, 2013.

[9] “TWT: Inquiry-Based Learning Strategy.” Accessed November 3, 2015. http://www.ndtwt.org/Blackboard/P2SST2/inqu.htm.

[10] “Khan Academy.” *Khan Academy*. Accessed November 30, 2015. http://www.khanacademy.org.