The Art of Fractals

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Contents of the curriculum unit

Overview:

The school that I teach in has a student population of about 2800. It is a collegiate like campus with 13 buildings on 62 acres of land. The composition of student body has an international flavor with over 34 countries represented in the 2008-2009 school year population. Myers Park embraces a diverse student population that is 59%white, 24% black, 6% Asian and 9% Latino and 2% other. (Data obtained from Myers Park High School website).

I teach AP Calculus AB, Calculus BC, IB Math Standard Level and IB Math Higher Level. We follow a hybrid schedule at Myers Park. That means some of our classes are 4x4, they meet every day for half of the year, and some are A/B day, they meet every other day for the entire year. Most of the AP and IB classes follow A/B day schedule. I have about 30 students in most of my classes.

The classroom culture is really diverse. I have students from China, Vietnam, Korea and India. My AP and IB classes take their final exam in the first week of May. Since the school year ends in June, we have about a month after the exams to cover the topics that students are interested in. During that time, we also let students do some discovery based learning and have some fun with Math. I am planning to include the study of Fractals in my IB Math Methods SL/HL class. This class is mostly comprised of 11th and 12th graders. I have this class on A/B day schedule, which means I see my students every other day. That would also mean that if we start a lesson during the class, my students can finish it for homework by the time I see them in next class. This would save the actual class time. So this would be a nice enrichment activity and I do not have to use all my class time to complete it. But for teachers not on the same schedule, they can adopt some parts of this lesson. They can also assign some of the activities for homework and give students about a week’s time to finish an activity.

Objectives

My aim is to introduce my students to the art side of math through this unit. Keeping my IB objectives in mind, which include some time of independent investigative mathematical work for students and developing a rationale behind it, the extension of some of the topics to the art side of Math by introducing fractals should work great.

Being in an IB class, my students are trained to pay attention to the mathematical notations and topics worldwide. Most of the students are quite motivated to do well. At this level, they have already finished Algebra 1, Algebra 2 and Geometry. IB Math Methods is a two year course at our school. The first year is pretty intense and we teach calculus all through this year, as it is one of the biggest and important topic of the IB syllabus. At the end of this year, students can also take AP Calculus AB exam as the entire content of the AP Calculus syllabus is covered during this year. The second year of Methods class continues with the study of topics like Algebra, Vectors, Statistics, Functions, Trigonometry and review of Calculus. During the second year, students are also responsible for some portfolio work based on mathematical investigation as well as mathematical modeling. Portfolio topics are usually based on practical real life concepts. Students are given a task and then a series of questions. As part of their investigation or modeling, they are responsible to answer those questions, find a pattern, come up with a mathematical process for the given situation and then discuss the scope and limitations of their findings. Most of this is independent work for students but we do a practice portfolio as part of our class work so they can get an idea of what is expected from them and then assign their independent tasks. I also assign some of this work as group projects so students can discuss with each other. Portfolios are part of their IB grade, so they take the work seriously. Also, for teachers this is the means of developing critical thinking skills and mathematical query in students. These portfolios are mostly based on real life problems on topics from the IB curriculum so students can see the mathematics and its use in everyday life.

The idea of Fractals came in my mind as part of the portfolio work so I can show my students the art side of mathematics. I can include some activities of this topic under the mathematical investigation part of their portfolio work. Since the rubric for portfolios require them to use technology, they can use computers to generate beautiful fractals. Also this is one of a topic of current mathematical research and it is so interesting to see creativity with mathematics just by the simple principle of self similarity in many cases.

This unit on Fractals will fit in after the end of first quarter, right after we finish sequences and series, including arithmetic and geometric series and iteration. Our first portfolio is also due around this time. Through the study of fractals, my students can see the math involved in art along with doing mathematical computations to get the principle behind the fractals. I will teach this complete unit in about 4 lessons and each lesson should take about 2 days. A lot of things that I am going to include in this unit are an extension of the syllabus so I can enhance my students' understanding. One may do as much as the time allows. Also some lessons could be assigned for homework, especially if we meet our students every other day instead of every day.

Rationale

Why should we bother about Fractals? This is not a new invention but I started paying attention to this topic only after taking this seminar. Fractals are a perfect combination of Math and Art. Fractals are objects that show self similarity. In our real life, in nature, we see many things that follow the principle of fractals but we do not necessarily notice them. Before I start the unit, I think it is important for my students to understand the vocabulary involved in each lesson. So I am going to introduce them to some basic terms like iteration- which means execution of a set of instructions for a given number of times, sequences- which involve identifying a pattern and working accordingly etc. Once they are aware of the terms, I can communicate with them in appropriate and standard terminology. As I am planning to work on fractals after finishing sequences and series, once I start this topic, my students will have the knowledge and idea of working with geometric and arithmetic sequences as well as iteration.

Fractals are a beautiful way of connecting mathematics to the world of art. I am a math person and not so artistic in nature. But this topic caught my attention as soon as I realized that the process of iteration can produce beautiful fractals. Many times as math teachers we get questions from students as to why are we studying this topic or where do I use this topic in my real life. We always talk about Newton's methods or Euler's methods and students feel like they are just learning the past. So the study of fractals relates or connects things together and it is a topic of the current mathematical research.

History of fractals goes back to 1975 when Benoit Mandelbrot observed that geometric objects like the cantor set and the Sierpinski triangle were not mathematical pathologies. Rather these complicated sets provided a geometry that is in many ways more natural than classical Euclidean geometry for describing intricate objects in nature such as coastlines and snowflakes. Thus was born fractal geometry. In Latin fractus means "broken" or "fractured".

A mathematical fractal arises from an equation that undergoes iteration or a form of recursion. A fractal follows a simple definition and is a subset of R^n which is self similar and whose fractal dimension exceeds its topological dimension. The use of computers and computer graphics has made it possible for people to see what fractals look like instead of envisioning them in their minds.

One nice thing about including this topic in my curriculum is that my students are not only learning math but it also makes them to pay attention to some of the objects that they see in their everyday life. Students will also find connections to various mathematical topics like geometric sequences, similarity, Euclidean geometry, logarithms, Pythagorean Theorem and process of iteration.

Since I am planning to cover this unit on fractals in my IB Math standard level class, there is not much work as far as basics are concerned. From their previous math classes, these students are familiar with the meaning and properties of logarithms, Pythagoras theorem, similar triangles and even similar figures. My class size is about 30 students which help me to give more individual attention to my students.

Strategies

This unit will include lecture, group work and independent work. I do believe that lecture is important if we are introducing a new topic but at the same time it is vital that students get some time for group work to share their findings and observations with each other. They learn more from their peers than just listening to the teacher for the entire time. For introducing the topic, I will lecture for a few minutes and then I will let them work together in cooperative groups. The benefit of working in groups is they share their understanding with each other and learn more from that and also group work helps a student who is otherwise having a hard time to visualize the topic. Also, I have promethean board in my classroom, so using technology I can show them construction of some fractals. This will create their interest in the topic. One such website that shows the construction of these beautiful fractals is from the National Library of Virtual Manipulatives, http://nlvm.usu.edu/en/nav/frames_asid_135_g_2_t_3.html. I am hoping that with my varied instructional techniques, I can reach all my students. For further details of strategies, see the next section with classroom activities.

Classroom activities

Before I start talking about fractals, there is some background work that needs to be done including introducing students to the terminology involved as well as covering the prerequisite topics for this unit. These lessons will also count towards the syllabus for this class.

For my **first lesson**, I will start talking about iteration. When we actually do fractals, we will need the knowledge of applying iterations. Iteration is the act of repeating a process. So I will teach them what iterations mean and how we apply iterations once or multiple number of times using the same mathematical formula or a mathematical process. We will work with iterations using very simple linear functions or a very simple concept so they get the idea of repeating the steps. During the same lesson, I will introduce sequences including arithmetic and geometric sequences. Working with sequences is part of the syllabus and students are also tested on this. At this point, I will make sure to point out the difference in the pattern of terms of arithmetic sequence compared to a geometric one. Arithmetic sequence has a common difference between its terms whereas a geometric sequence has a common ratio between its terms. We will recognize the pattern and find the general term or the stated term and vice versa using the formula for nth term for both of these sequences.

To find the nth term of an arithmetic sequence, we will use the formula

$a_n = a_1+(n-1)d$,
where a_1 is the first term of the sequence and d is the common difference between its terms.

To find the nth term of a geometric sequence, we will use the formula

$a_n = a_1 r^{(n-1)}$,
where a_1 is the first term of the sequence and r is the common ratio.

I will also talk about the difference in sequences and series and then work on a few examples to find the sum of finite arithmetic and geometric series as well as the sum of infinite geometric series.
To find the sum of a finite arithmetic series, we will use the formula

$S_n = n/2\ [2a_1+(n-1)d]$

To find the sum of a finite geometric series, we will use the formula

$S_n = \dfrac{a_1(1-r^n)}{(1-r)}$ and

To find the sum of an infinite geometric series, the following formula can be used
$S = \dfrac{a_1}{(1-r)}$

They will have some time to work independently on sequences and series and to deal with a variety of problems on this topic.

Exercise on Arithmetic and Geometric sequences and series

1) List the first three terms of the following sequences

   $U_n = 2^{n-1}$

   $U_n = \dfrac{2n+1}{n+2}$

2) A sequence is defined by $U_n = 8-5n$

   a) Prove that the sequence is arithmetic.

   b) Find the first term and the common difference

   c) Find the 50^{th} term of the sequence

   d) Find the sum of the first 50 terms.

3) Show that the sequence 7,14,28,56,… is geometric. Find its general term U_n and hence find U_{10}

4) Find k if 2k, k-2 and k+7 are the consecutive terms of an arithmetic sequence.

5) Find k given that 3, k, k^2-8 are the consecutive terms of a geometric sequence.

6) Find the general term of an arithmetic sequence given that U_7=21 and U_{10} = 3. Hence find the value of U_{45}.

7) A sequence is defined by = $3\left(\dfrac{1}{2}\right)^{n-1}$

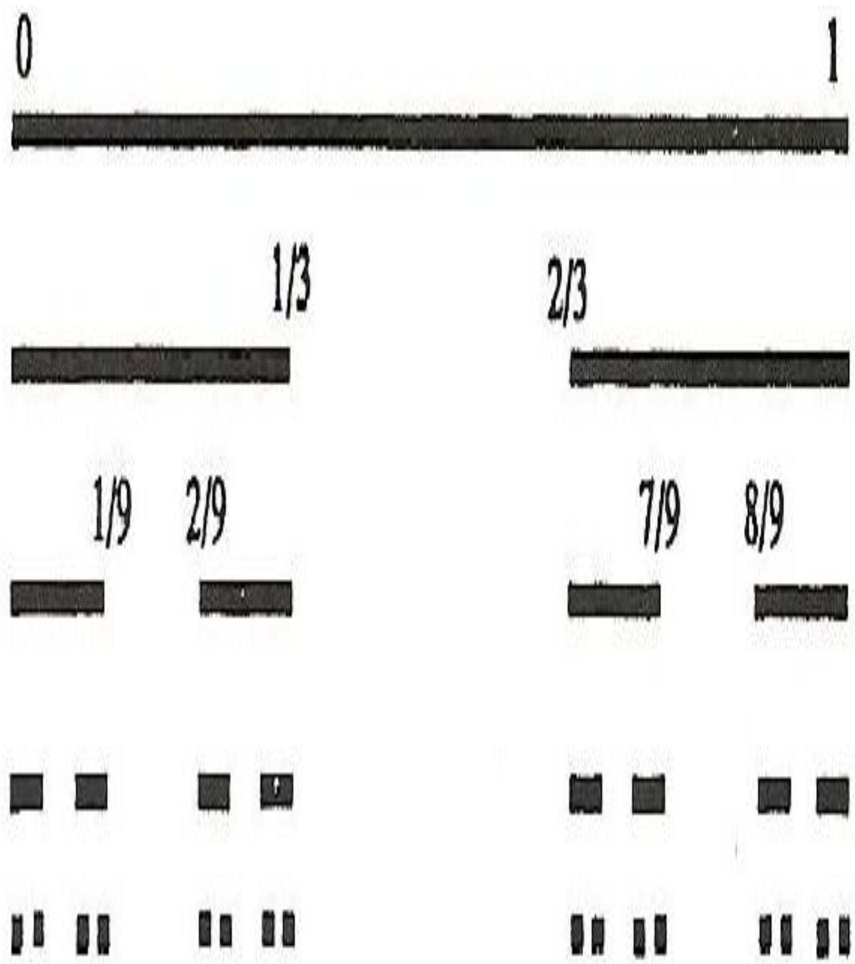
   a) Prove that the sequence is geometric

b) Find its first term and common ratio, r.

c) Find the 15th term of the sequence.

8) Find the general term of a geometric sequence given that its fifth term, $U_5 = 21$ and its eighth term, $U_8 = 168$.

For my **second lesson**, I will introduce the concept of fractals through very simple examples. A fractal is an object possessing the property of proper self similarity. From their previous geometry class, they are expected to have the working knowledge of similarity. The simplest fractal to introduce is a Cantor Set as this is the most basic kind of fractal. This is also called Cantor middle third set. To construct this set, we begin with the closed interval $0 \le x \le 1$. From this we remove the middle third, the open interval $\frac{1}{3} < x < \frac{2}{3}$. We do not touch the end points and what remains is a pair of closed intervals, each of them one third as long as the original. So we are left with two closed intervals, $0 \le x \le \frac{1}{3}$ and $\frac{2}{3} \le x \le 1$. Then we do this again. From each of the remaining intervals we remove the middle third. So, from the interval $0 \le x \le \frac{1}{3}$, we remove $\frac{1}{9} < x < \frac{2}{9}$ and from $\frac{2}{3} \le x \le 1$, we remove $\frac{7}{9} < x < \frac{8}{9}$. In each case, we leave behind the end points, now four closed intervals remain, each one-ninth as long as the original interval. We repeat this process over and over. At each stage, we remove the middle third of each of the intervals remaining at the previous stage. This process is continued indefinitely and the points left in the set are called the Cantor middle-thirds set.



For my **third lesson**, we will work on another important and amazing fractal called The Sierpinski Triangle. This was introduced by the great mathematician Waclaw Sierpinski (1882-1969) in the year 1916. Just like the Cantor Set, this object may as well be obtained by an infinite sequence of "removals". To construct this fractal, we start with a simple equilateral triangle with sides of length one unit. Length of the sides of triangle can be varied but calculations will be simpler if we start with one unit to begin with. This triangle can be imagined as a solid object. We connect the middle points of the three sides of this triangle and get four smaller equivalent triangles. From here, we remove the middle equilateral triangle. Now we are left with three smaller equilateral triangles with sides of length $\left(\frac{1}{2}\right)$. In the next step, we repeat the same process with the three remaining triangles to obtain 9 equilateral triangles with sides of length $\left(\frac{1}{2}\right)^2$. If we continue this construction, then at the nth stage we will have 3^n equilateral triangles with sides of length $\left(\frac{1}{2}\right)^n$. Like the Cantor Set, the Sierpinski Triangle possesses the property of self similarity. The construction of this basic Sierpinski Triangle is detailed in the Appendix.

The Sierpinski Triangle can be constructed in many different ways. We do not necessarily have to start with a triangle. One such method is described in Appendix **3**. Here, instead of starting with a triangle, we start with a square. This is called Initiator. Our process of removal is mentioned in the figure in step 1 called the generator. Then we repeat this process, described in generator, and we continue the process infinitely.

As an extension of this construction, Sierpinski triangle can be constructed from Pascal's triangle. I will assign this for homework but one may do this activity as part of the class work as well. Before assigning this activity I will make sure to discuss the construction of Pascal's triangle with my students. Pascal's triangle gives us the coefficients of Binomial expansions. Its first row starts with the number 1 and is considered step 0. The next row with step 1 consists of numbers 1 and 1. The beginning and ending numbers in each row are always 1 and the middle numbers are the addition of the two numbers to the left and the right in the previous row. We can continue this process to construct Pascal's triangle. Here the construction of Pascal's triangle is shown to step 10.

```
                    1
                  1   1
                 1  2  1
                1  3  3  1
              1  4  6  4  1
            1  5  10  10  5  1
          1  6  15  20  15  6  1
        1  7  21  35  35  21  7  1
      1  8  28  56  70  56  28  8  1
    1  9  36  84  126  126  84  36  9  1
  1  10  45  120  210  252  210  120  45  10  1
```

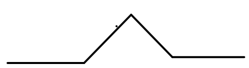
In order to construct Sierpinski triangle from Pascal's triangle, we can draw a grid circle around each integer in the Pascal's triangle. If the integer is an odd number, we color the grid and if the integer is an even number, we leave the grid uncolored. The result is shown in the Appendix.

For my **fourth lesson**, I will work on another common fractal called the Koch Snowflake. This definitely comes under the IB Portfolio section for my students. Unlike the Sierpinski triangle, the Koch snowflake is generated by an infinite succession of additions. It is amazing to see how a simple equilateral triangle transforms in a complicated snowflake like structure. To construct this piece of art, we start with an equilateral triangle of side length 1. For our first step in the process, we remove the middle third of each side of the triangle, just like we did in the construction of the Cantor middle - thirds set. This time, however, we replace each of these pieces with two pieces of equal length, giving it a star shaped region. The new figure has twelve sides, each of length $\left(\frac{1}{3}\right)$ .Now we repeat this process. From each of these sides we remove the middle third and replace it with a triangular “bulge” made of two pieces of length $\left(\frac{1}{9}\right)$. This process is continued over and over and the ultimate result is a curve that is “infinitely wiggly”. There are no straight lines in it whatsoever. This object is called the Koch Snowflake. If we magnify and look at pieces of the snowflake, we can see the self similarity. At each stage of the construction of the Koch curve, magnification by a factor of 3 yields the previous image. This means that the ultimate figure is self similar or each part of the figure is similar to each part of the original figure.

While working on Koch Snowflake, it is important to remind our students that even though the perimeter of the figure goes to infinity from the first step to the point where we apply iterations a number of times, yet the area remains finite. This is the beauty of this fractal.

In order to interpret our understanding mathematically, I will require for my students to establish a relation between number of sides at each stage, length of the side, perimeter of the snowflake as well as its area.

It is important to notice that each side of the triangle in our step one, or each straight line is replaced by the shape



That means, each side ends up being four sides. The number of sides at any stage n, can be given by

$N_n = 3 \times 4^n$

As we also noticed that the length of the next side is one-third of the previous length,

$L_n = \left(\frac{1}{3}\right)^n$

The perimeter of the Koch snowflake is given by number of sides multiplied by the length of one side. Using the above two formulas, we can say that

$P_n = 3 \times \left(\frac{4}{3}\right)^n$

Likewise, an expression for area can be derived based on the sum formula for geometric progression.

If we see the construction of snowflake, we can see that area at stage one is given by

$A_1 = A_0 +$ area of new triangles

As the linear scale factor is one-third the area scale factor is one-ninth. So each additional triangle at the new stage has an area of one-ninth the area of each triangle at the previous stage.

To get the total area we need the number of triangles that have been added to the Koch snowflake.

$$A_1 = A_0 + \text{new triangles x } \frac{1}{9} \text{ x } A_0$$

$$A_1 = A_0 \, [1 + \text{new triangles x} \frac{1}{9}]$$

$$A_1 = \frac{\sqrt{3}}{4}\left(1+3\times\frac{1}{9}\right)$$

$$A_1 = \frac{\sqrt{3}}{4}\left(1+\frac{3}{9}\right)$$

At stage two, each new triangle is one-ninth at stage one, or 1/9 x 1/9 = 1/81 of the triangle at stage 0. At stage 2 there are 12 new triangles.

$$A_2 = A_1 + \text{new Area}$$

$$A_2 = A_1 + 12\times\frac{1}{81}\times\frac{\sqrt{3}}{4}$$

$$A_2 = \frac{\sqrt{3}}{4}\left(1+\frac{1}{3}\right)+12\times\frac{1}{81}\times\frac{\sqrt{3}}{4}$$

$$A_2 = \frac{\sqrt{3}}{4}\left(1+\frac{3}{9}+\frac{12}{81}\right)$$

The number of extra triangle is the same as the number of sides from the previous stage. So for the nth stage, that will be $3\times\left(\frac{4}{3}\right)^{n-1}$

We can see that this follows the geometric pattern, so we can use the formula

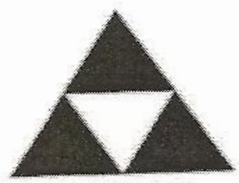
$$S_n = \frac{a_1(1-r^n)}{(1-r)}$$

to evaluate the geometric progression.

$$A_n = \frac{\sqrt{3}}{4}\left(1+\frac{3\left(1-\left[\frac{4}{9}\right]^n\right)}{5}\right)$$

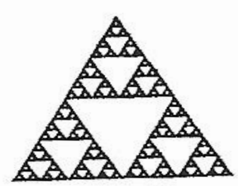
Construction of the Sierpinski triangle

Side length = 1
Step 0: Initiator



3^1 triangles: Side length = 1/2
Step1: Generator

3^2 triangles: Side length = (1/2)^2
Step 2



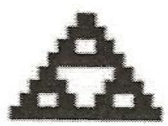
3^5 triangles: Side length = (1/2)^5
Step 5

Another construction of Sierpinski triangle

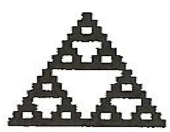
Step 0: Initiator

Step 1: Generator

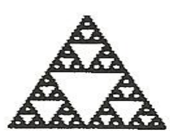
Step 2



Step 3



Step 4



Step 5

Sierpinski triangle from Pascal's triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

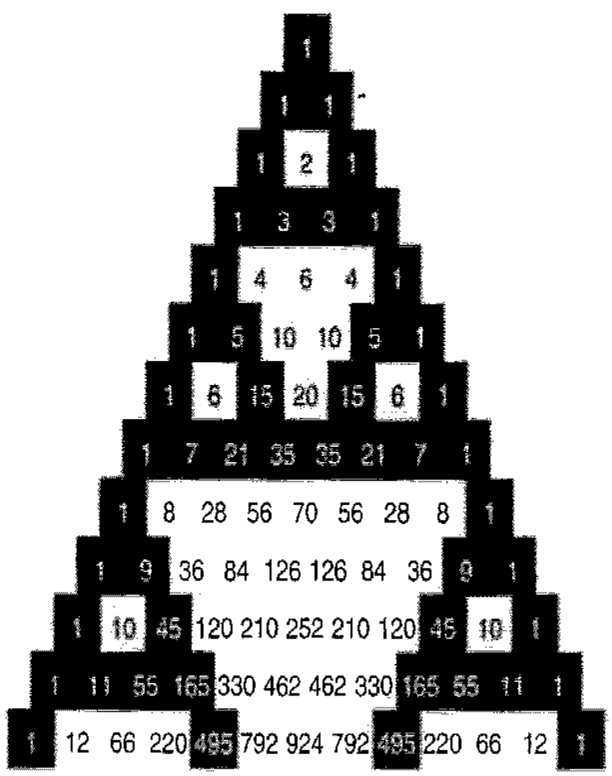
1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

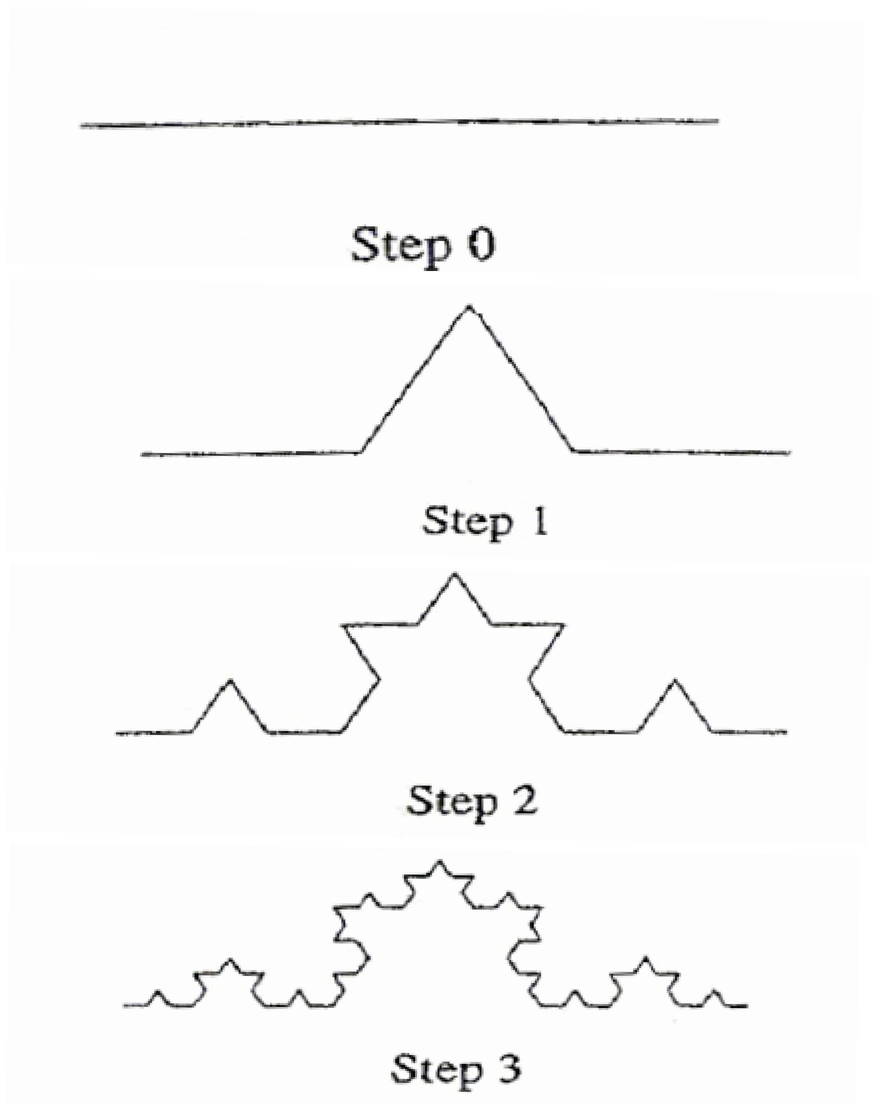
1 10 45 120 210 252 210 120 45 10 1

1 11 55 165 330 462 462 330 165 55 11 1

1 12 66 220 495 792 924 792 495 220 66 12 1



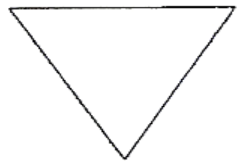
Construction of Koch curve



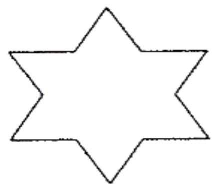
Step 4

Step 5

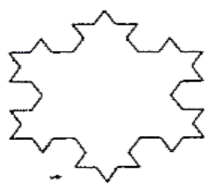
Construction of Koch snowflake



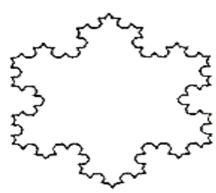
Step 0



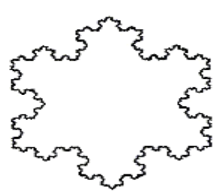
Step 1: Generator



Step 2



Step 3

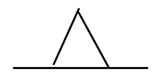


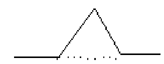
Step 4

Worksheet on Koch Snowflake for students (Portfolio task)

Investigation on Von Koch's Snowflake Curve

To draw Von Koch's Snowflake Curve we

- Start with an equilateral triangle, C1
- Then divide each side into 3 equal parts __,__,__
- Then on each middle part draw an equilateral triangle 



- Then delete the side of the smaller triangle which lies on C_1
- The resulting curve is C_2, and C_3, C_4, C_5 . . . . are found by 'pushing out' equilateral triangles on each edge of the previous curve as we did with C_1 to get C_2.

We get a sequence of special curves C_1, C_2, C_3, C_4 . . . and Von Koch's curve is the limiting case, i.e., when n is infinitely large for the sequence.

Your task is to investigate the perimeter and area of Van Koch's curve.

What to do:

1 Suppose C_1 has a perimeter of 3 units. Find the perimeter of C_2, C_3, C_4, and C_5.

(**Hint:** __,__,__ becomes  i.e., 3 parts become 4 parts.)

Remembering the Von Koch's curve is C_n, where n in infinitely large, find the perimeter of Van Koch's curve.

2 Suppose the area of C_1 is 1 unit^2. Explain why the areas of C_2, C_3, C_4, and C_5 are

$A_2 = 1 + \frac{1}{3}$ units ^{2} $\qquad$ $A_3 = 1 + \frac{1}{3}\,[1 + 4/9]$ units^2

$A_4 = 1 + \frac{1}{3}\,[1 + 4/9 + (4/9)^2]$ $\qquad$ $A_5 = 1 + \frac{1}{3}[1 + 4/9 + (4/9)^2 + (4/9)^3\,]$ units^2 .

Use your calculator to find A_n, where n = 1, 2, 3, 4, 5, 6, 7, etc., giving answers which are as accurate as your calculator permits.

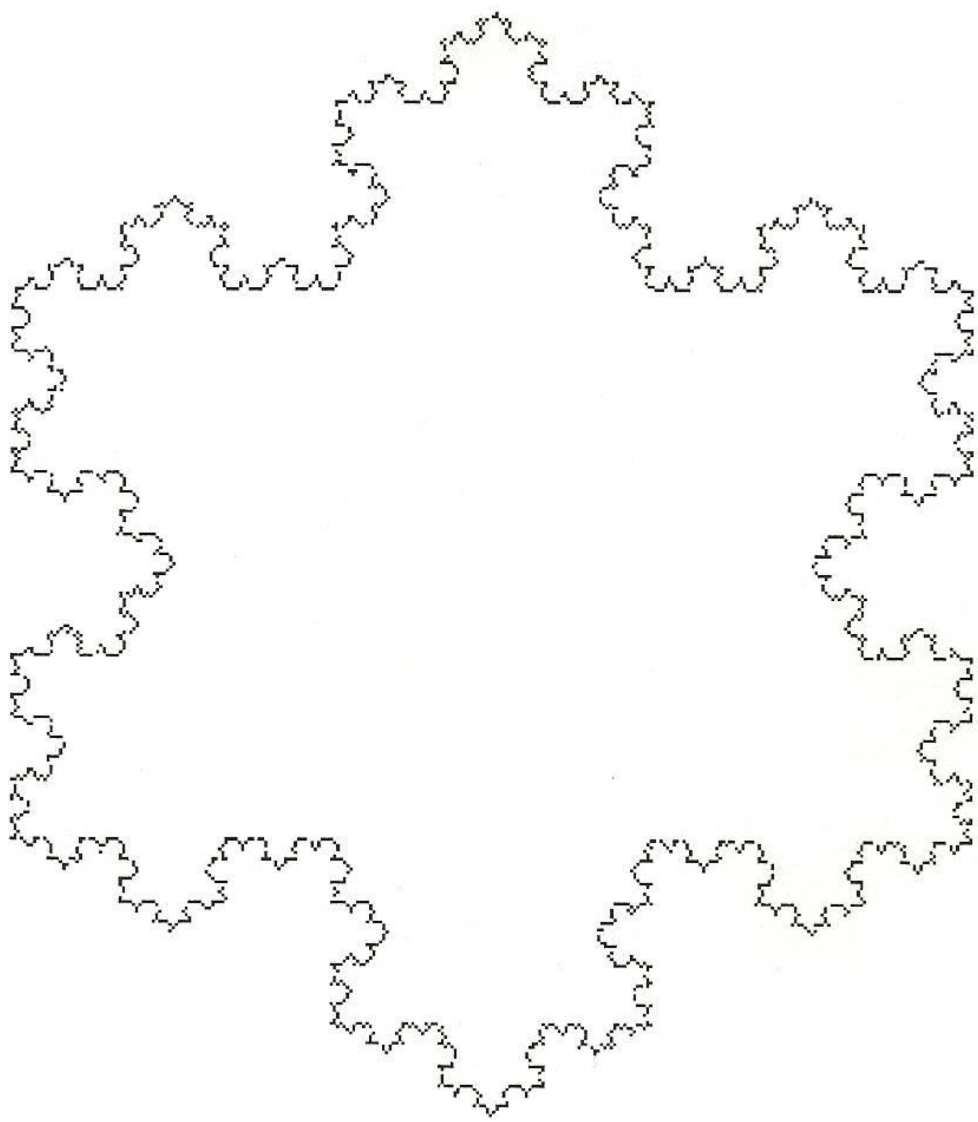
What do you think will be the area within Von Koch's snowflake curve?

3 Similarly, investigate the sequence of curves obtained by pushing out squares on successive curves from the middle third of each side.

Koch Snowflake

HL Portfolio Assignment Type

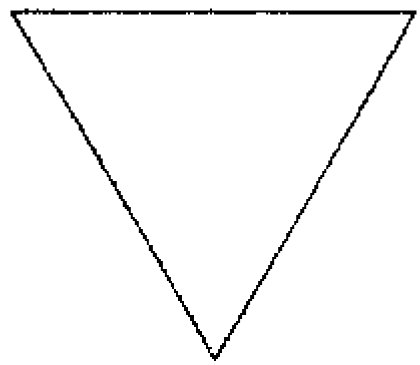
Solution

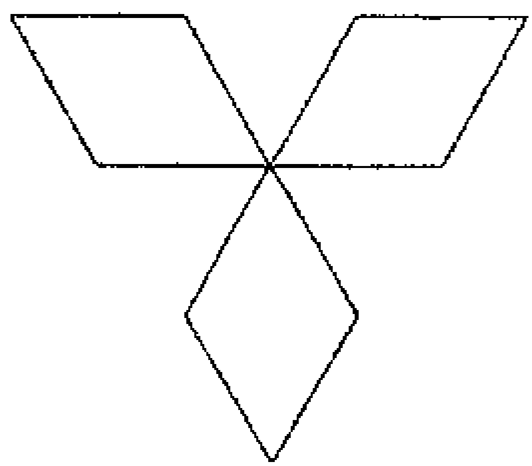


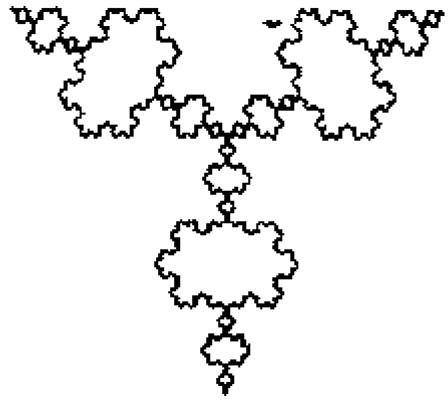
More Practice on Fractals

For the following pictures, the initiator and the generator of a fractal are provided. Use five iterations to generate the picture shown.

1







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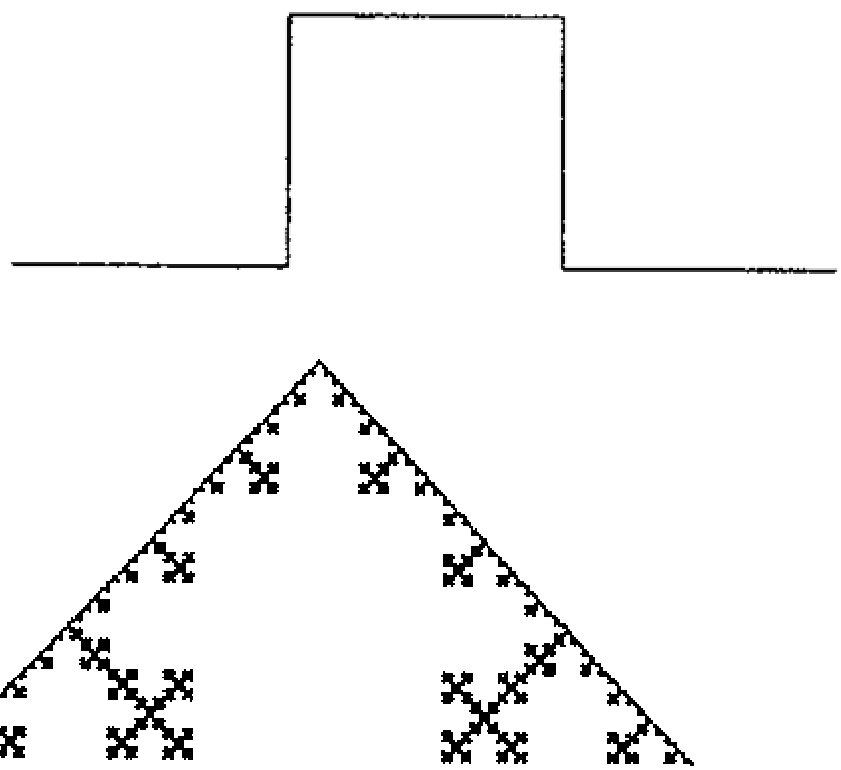

Works Cited

Devaney, Robert L.. "The Quadratic family, Fractals." In *A first course in chaotic dynamical systems: Theory and experiment*. Reading, Massachusetts: Addison-Wesley, 1992. 75-76, 182-184.

The topics that I used from this book are,The Cantor set revisited, The Koch Snowflake. Both of these topics are explained really well supported by appropriate figures.

Elaydi, Saber. "Fractals." In *Advances in discrete dynamical systems* . Tokyo: Mathematical Society of Japan, 2009. 289-291.

The Sierpinski Triangle has been explained very well in this book. The terminology is user friendly and easy to follow. I also used the pictures of the Sierpinski triangle from this book.

Owen, John. "Sequences & Series." In *Mathematics for the international student: mathematics SL*. Adelaide Airport, S. Aust.: Haese & Harris, 2004. 58-59.

I use this book for my IB class. IN this curriculum unit, I have used the assignment for Koch Snowflake and some of my exercise problems are also inspired from the review exercise of arithmetic and geometric sequences and series.

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