Teaching the Concept of One as It Applies to Percents

Michael Pillsbury

Introduction

The middle years of education offer unique challenges to teachers. Students are experiencing adolescent development and their bodies are changing as much as their minds and views on life. In mathematics moving students from “concrete” thinking to “abstract” is critical to the success of each individual student. Students cannot physically handle or touch things and see the answers. They have to stretch their mind beyond what they see in the classroom and envision the outside world and the applications of mathematics in that world without actually being there. One of the most crucial concepts of mathematics that students must master is percents. I am a teacher who is beginning his sixth year of teaching. I have struggled to get students to truly understand the concept of percents. It is fairly simple to get them to do the mechanics of problems or plug in numbers into a formula to get an answer. It is a different concern to get them to explain in words (writing) what they are doing and what their answer actually means. Because they lack a depth of understanding of the content when they are faced with a problem that has been altered from the format they are accustomed to they often struggle. My plan to teach students a higher level of understanding was to ramp up to percents by linking to prior learning of fractions and decimals so that students could grasp what percents actually are. The overarching theme of the unit is “one.” This describes the concept of one whole, one hundred percent 1/1 or 1.00. The “one” also defines the standard by which other things are measured. I have chosen to use the term “one” instead of whole as many textbooks and teachers prefer to use. There are many scenarios that involve situations where defining the part and the whole does not make sense such as growth of the same object.

Objectives

The following objectives are part of the North Carolina Standard Course of Studies:

1. Develop and use ratios, proportions, and percents to solve problems.

1.03 Develop flexibility in solving problems by selecting strategies and using mental computation, estimation, calculators or computers, and paper and pencil.

The National Council for Teachers of Mathematics objectives would be addressed by having students understand numbers and be able to work flexibly with fractions, decimals and percents to solve problems as well as develop meaning for percents greater than one hundred and less than one.

Upon completion of the unit students should be able to identify the relationship of fractions, decimals and percents. Students should understand the concept of fractions in greater detail and by using the concept of one be able to explain why when adding or subtracting fractions a common denominator is necessary and why the denominator is never added or subtracted. Students should have an understanding of place value as it relates to not only our own number ten based system but also other number based systems. Students should be able to solve problems involving percents to include percent of a number, percent decrease or increase as well as common sales problems so prevalent in summative assessments.

Rationale

Students struggle with the concept of percent. When I teach percents to students I feel as if they think it is a concept that is greatly different from what they already know. Students' understanding and knowledge of percents, especially if it is their first exposure, is critical to understanding numbers. Often this unit is the first time that there is a solid connection between the relationships of fractions to decimals. It is vitally important that the triad of percents, fractions and decimals come together in a clear understanding of relationships.

Percents are the culmination of understanding of many topics. A percent is simply the ratio of a number to another number expressed to the 100^{th} place value. An example is ½ the ratio of 1 to 2 is 0.50 which when taken to the 100^{th} place value the value is 50 %. Students should understand that our number system is based on the number ten and therefore each decimal spot to the right of the decimal represents a power of ten. A great time to explain this might be earlier when exponents and scientific functions are being taught. Students should see that is actually 1/10 or .10 and of course is 1/100 or .01. To understand the concept of percents I have decided to design my unit by first discussing fractions so that students have a true grasp of the concept. I then move on to converting fractions to decimals and the concept of decimals and place value. Understanding the decimal representation of a number is impossible if the place value is not understood. The progression continues to the last concept of what a percent actually is and its definition. How the percent relates to the number one and how it is determined is also covered.

Percents dominate a large part of our life and are present everywhere. Real world applications of percents are important to conceptual understanding and allow for great problem solving opportunities. A unit on percents is great for not only real world problems but interdisciplinary opportunities as well. Reading and writing in other content areas is critical to learning and problem solving with percents facilitates reading comprehension and when done correctly allows for rich writing opportunities.

Our school is an International Baccalaureate school and as such it is important to include various areas of interaction in instructional units. By nature percents allow for great problems that can involve community and service, human ingenuity, environmental issues and health. Giving context to problems will allow for a much richer learning experience.

Unit Design

Fractions

The unit will begin with a review of fractions. Teachers should be mindful not to concentrate on the numbers here but the concept. I have found that pizza fractions work well. It provides students with a circle and thus can be translated to percents very easily. All students have been exposed to pizza and how pizzas are cut into smaller segments. These segments represent fractional portions of the pizza itself. This first section is where the concept of "one" should be introduced. The pizza itself is the "one." It represents one whole and this of course can be shown as 1/1 or as other fractions depending on the number of slices the pizza is cut into. If it is cut into eight slices then it is 8/8 or into four slices 4/4. One critical concept that must be focused on is that each slice has to be equal in size. All of these equal sections combined form the "one" pizza.

Fractional representations of the pizza where the numerator is less than the denominator mean that less than the "one" pizza is being discussed. This concept is important. Students must recognize that if the numerator is less than the denominator the fraction represents less than one. This idea can obviously be built upon to the next step of greater than one. Here a whole pizza with another slice from another pizza of equal size might be discussed giving us 9/8. What this represents is vitally important to student understanding. What the nine represents and what the eight represents must be discussed. The eight represents the "one" or how many slices make up a whole pizza. The nine represents the amount of slices we have. The conclusion that should be reached is that if the numerator is great than the denominator then there is more than one present. Discussing what would it mean if we had written 9/9 when we had one pizza and the extra slice? This obviously would have meant that the concept of the "one" had changed. The new definition of "one" would be a pizza and a slice. This does not make sense so that is why the fraction is written as 9/8 to keep the definition of how many slices constitute the "one" at eight.

The Singapore Math Model Method utilizes bars to show the relationship to the whole or one. By simply drawing a bar and defining the size of a bar a fractional representation can be made. The basic concept works well with numbers that are factors but can be less clear if the bar length is not a multiple or factor of the denominator.

Example:

The above indicates a top bar that could have a value of 6 or any multiple or factor of 6 while the bottom bar would reflect a fractional representation of the top bar. For example the bottom bar could be 4/6 if the top bar had a value of 6/6 which reduces to 2/3. If the top bar were 3/3 then the bottom bar would still be 2/3. The one is defined by the number of sections the bar is divided into. If the top bar were 2/2 then we have a more difficult problem because the bottom bar would be ½ plus 1/3 of the ½ of the rest of the bar. Students would then have to multiply 1/3 and ½ to get 1/6 and then add that to the ½ they have which becomes 3/6 plus 1/6 which is again 4/6 or 2/3.

A problem with the above bar representations does not work out very nicely. The bars do not match up and the 6 and the five pieces do not go together very nicely. The solution to this of course is to divide the top bar into 30 segments then the below bar could be matched up with the above bar. Showing this would help students understand why it is necessary to have a common denominator when adding or subtracting fractions. It is a useful tool for comparing them as well!

Converting fractions to decimals

The next step is to show how fractions can be turned into decimals. This is great practice for long division which is a skill many students lack. Students should recognize that fractions are in fact simply division problems. ½ can be written as 1÷2 and it means the same thing. Of course an infinite number of fractions would result in the same decimal equivalent. Students must understand that is equivalent to . As long as the quotient is the same when the numerator is divided by the denominator then the fractions are equivalent. Introducing the concept of ratios by using equivalent fractions demonstrates that no matter how big the numbers if they are proportional ratios then in simplest from they are same value. If I have 1,000,000 boys and 2,000,000 girls then the ratio is still simply 1 boy to every 2 girls. Students should then solve the problem through long division. The resulting answer of 0.5 is an illustration of how a fraction can be turned into a decimal. In other words when two real numbers are divided the quotient will be the decimal representation of the ratio of those two numbers.

Students should be reminded of their lessons in elementary school where a remainder was written as R something. At their current level of mathematics the remainder will be expressed differently. Two examples of what may be focused on here are how to solve 7/3 two ways, through long division and fractionally. With long division the answer is   and fractionally the answer is 2 1/3. Both are accurate representations of the answer. What they mean should be discussed. The main idea is that there is a little more than two wholes present. Perhaps if we refer back to the pizza example we could say that there are two pizzas made up of three slices each and an extra slice. This might give a visual representation to some students. Bring this back to the "one." What is the one here? Of course it is 3/3. Dividing integers is not difficult but the relationship of the answer to the fraction is critical. A decimal that is less than one would have a fractional representation where the numerator is less than the denominator. A decimal that is greater than one would have a fractional representation where the numerator was greater than the denominator. Getting students to understand this is an important concept.

Decimals

We can think of the decimal as an address on a number line. Roger Howe indicated that decimals can be thought of as specific addresses. For example 0.375 can first be thought of as being between 0 and 1 on a number line because in the ones place there is a zero and then there are numbers following greater than zero.

0 .1 .2 .3 .4 .5 .6 .7 .8 .9 1

The first address between 0 and 1 we identify is 0.3 so the decimal has to be between 0.3 and 0.4. If the view is magnified then the area between 0.3 and 0.4 has to be looked at for the address.

.30 .31 .32 .33 .34 .35 .36 .37 .38 .39 .40

The second address is between .30 and .40 which is magnified above. More specifically it is between 0.37 and 0.38 so the number line must be magnified again.

.37 .371 .372 .373 .374 .375 .376 .377 .378 .379 .38

The final address is now located. Students should realize that all number addresses found between 0 and 1 will be percentages less than one hundred percent. Number addresses that are above 1 and before 2 will be one hundred and something percent and so on.

Place values must then be discussed. A great way to discuss place values is by using exploding dots. Jim Tanton of the St. Marks School created exploding dots and it is a creative and fun way to explore place value. The concept is simple. There is a machine that has chambers in it. The idea is that the machine has some numerical value assigned such that when a certain number of dots exist in one box they explode and a dot occurs in an adjacent box. One example comes from our own decimal system.

If ten dots go into the machine in the first box then they explode from the first box and one dot appears in the second box. Having the students first do this with a machine that explodes when two dots are in the same box and the result is one dot in the next box to the left will lead to the discovery when written there is no number above one. The teacher should explain that this is binary or the base two number system. Explore various numbers systems and then return to base ten number system. I have included an example below illustrating how the number ten can be arrived at. If ten dots are put into the first box they explode and then one dot appears in the second box. This actually represents the number ten. Students should then explore this with various examples.

1. 0

Use exponential representation to show why. is ten, is one, and is one tenth. Written in decimal form they are in order 10, 1 and .1 in fractional form they are in order 10/1, 10/10 and 1/10. Write above the box the exponential value of each box.

Try to give students examples and have them write out the value of the number. Do this with negative exponents included so that they can see how numbers to the right of the decimal point are determined. The obvious place value of importance here is the one hundredth place. Define decimal and discuss what a decimal point is. The fact that decimal means base ten is helpful in defining the value of each place. The decimal point is a separator to indicate a fractional part of the number. When students have decimals they should be able to identify the place of each number. For example .342678 would have the 3 in the tenth's place, the 4 in the hundredths' place, the 2 in the thousandth's place, the 6 in the ten thousandth's place, the 7 in the hundred thousandth's place and the 8 in the millionth's place. The pattern should be discussed here and referring back to exponents may help. Many students will be confused as to why there is no oneth's place. If we can represent a number in decimal form then expressing it as a percent is a simple task. Exploring decimals here is important. If a number is multiplied by ten what happens to it? The fact that the decimal moves one spot to the right is a key concept. Why does it move one spot to the right? What if the number is multiplied by one hundred or a thousand how many spots to the right does the decimal move? This can be explained by referring back to the place to the right of the decimal. Examples can be provided with whole numbers or real numbers. Show that 3 multiplied by ten is 30 and 3 multiplied by one hundred is 300. The decimal spot is simply moving to the right a certain number of times based on the power of ten the number is being multiplied by.

Defining place value can also be done with addition utilizing the powers of ten and the commutative and associative rule for addition as well as the distributive rule Roger Howe explains in his essay that there is a very easy way to show students what they already do in a different format which shows clearly the idea of place values. An example might be as follows:

12,678 + 3,406 is the problem given.

So we rewrite this as (1 x 10,000 + 2 x 1,000 + 6 x 100 + 7 x 10 + 8 x 1) + (3 x 1,000 + 4 x 100 + 6 x 1).

The numbers here can be regrouped to be (1 x 10,000) + (5 x 1,000) + (10 x 100) + (7 x 10) + (14 x1).

This of course gives us:

10,000 + 5,000+ 1,000 + 70 + 14 which has to be broken down to
10,000 + 6,000 + 70 + 10 + 4
10,000 + 6,000 + 80 + 4

Which totals 16,084 using base 10

What if a number is divided by a power of ten? What happens to the decimal point? Show that 300 divided by ten is 30 or 300 divided by 100 is 3. The decimal moves to the left the power of ten that the number is being divided by. Roger Howe defines a decimal fraction as any fraction whose numerator is a non negative integer and whose denominator is a power of 10 and *a and m* are not negative. For example:

All numbers are not decimal fractions.
Percents

Next the concept of percent should be introduced. I have found the cent part of the word a good focus. We all know one cent is 1/100 of a dollar. We also know that when we say for example 55 Miles/ hour we say per when we see the fraction bar. So presto change per-cent is /100. The focus on one hundred is vital to understanding percents.

Percent means any number over one hundred or divided by one hundred. Moving away from over should be done as soon as possible. At a certain level of math division should be shown as which means five divide by six. A fraction is simply division and students must not see it as anything else. When discussing percents the fraction of a number divided by one hundred indicates the number we call percent. An example would be which is sixty five percent (65%). The concept of "one" should then be discussed. In terms of percents the "one" is obviously one hundred. If the numerator is less than one hundred it is less than one hundred percent. If the numerator is greater than one hundred it is greater than one hundred percent. Look at which is 165%. Key concept here is that when ever the numerator can be written over the denominator of one hundred we know our percent without any problem. This does not always happen and we have fractions that can be expressed in terms of the numerator over one hundred very easily.

If we had a pizza composed of eight pieces and there were only three pieces remaining what percent of pizza is remaining? The first step is to identify what is the "one" or the whole in the problem. Obviously a whole pizza consists of eight pieces so eight is the whole or the "one." The three represents a part of the whole. The fraction represents what we have left of our pizza. We have to refer back to converting fractions into decimals by recognizing our fraction is simply division we divide three by eight and get 0.375. This is critical to understanding. The decimal 0.375 is actually but we want to write it as a numerator over one hundred. We have to recognize what value is in the hundredths place. In this example it is 7. The decimal must be moved to reflect a fraction divided by one hundred in order to discover our percent. To move the decimal to the right two places we multiply by one hundred. The resulting 37.5 is our percent! Students should realize that once a fraction (division statement) is expressed as a decimal it can then be multiplied by one hundred to express it as a percent. This concept should be practiced so that students are comfortable expressing different representations such as decimals and division statements as percents.

Problems involving Percents

If the content of the unit has been understood by the students then moving into the problem solving stage is logical and should be easy. The first type of problem that should be discussed is finding the percent of a number. Students who have understood the content will realize that they have already been doing this! Asking a student to determine what percent of eight is three is set up as the aforementioned problem. The concept that needs to be understood here is what is the whole or the “one?” If the question is three is what percent of eight then the eight is the “one” and must be placed in the denominator. If the reverse question were posed eight is what percent of three then the “one” is the three. It is important that students understand early on that it is not always the larger number in the denominator and the smaller in the numerator. The focus of the lesson should be the meaning of the numbers not the size in determining their placement either in the numerator or denominator.

The Singapore Model Method

The Singapore Model Method utilizes the same bars as used with fractions. The model is described as the Part-Whole Model.

A bar is drawn to represent a whole and assigned a value, whatever the whole is. In my unit remember the whole is called the one. If for instance the whole is 500 and the question was to find 40% of five hundred then the whole or 500 is divided by 100 and the resulting quotient is multiplied by 40 so 40% of 500 is 200. The opposite is true as well if the 40% of the whole number is 200 then 200 is divided by 40 and the quotient is multiplied by 100.

The Comparison-Model compares two bars and one of the two bars is designated as the 100% or the one as I like to refer to it. The really interesting observation in the Singapore Model Method is when comparing bars the obvious is not true.

If bar A is 80% of bar B then that does not mean when comparing bar B to bar A that bar B is 120% of bar A. In fact bar B is 125% of bar A. This can be investigated by looking at the fractional values of comparing the bars. If bar A is 80% of bar B then fractionally this translates to 4/5 which is .8 or 80%. If we look at comparing bar B to bar A where bar A is the one (which means it is in the denominator) then the ratio is 5/4 which is 1 ¼ or 1.25 or 125%.

The next type of problem should be when the “one” is given and the part is unknown but the percent that the part is of the “one” is known. An example of this might be that if 75% of the pizza is remaining and there were eight pieces of pizza to begin with then how many slices are there? Setting up this problem is the key. We know that the “one” is eight and we have an unknown which we can represent by the variable X which represents the number of slices left. Based on our previous work we can represent the part over the “one” as a division statement and write is as a fraction X/8. This of course is equal to the 75% which based on previous work we know equals 75/100 or 0.75. Students must realize that they need to work with either the fraction or the decimal when solving this problem. By multiplying both sides by eight the answer is 8(.75) = 6 or 8 (75/100) which equals 600/100 which reduces to six. The answer is of course that there are six pieces of pizza remaining.

The next type of problem is when the part of the “one” is given and the percent it makes up of the “one.” The “one” is not given. an example might be if there are 5 slices of pizza left and this represents 62.5 % of the total pizza that we started with, how many slices were present before we ate some? The variable here is the “one” so the X would go in the denominator and the five would be in the numerator. The fraction of course is equal to 62.5% which must be written as either 0.625 or 62.5/100. If we solve the problem by first multiplying both sides by X and then dividing both sides by 0.625 we get the answer of eight.

If students become comfortable with solving the three different types of problems involving percents then the final critical skill to be developed is being able to identify the “one.” A great example is growth of an object such as a tree. If a tree was 35 feet tall last year and it grew 20% how tall is it this year? The students should first identify the “one” in this problem. Is it the height of this year’s tree or last year’s tree? When given thought the students should realize that this year’s tree is 20% taller than last year’s tree therefore it represents 120% and last year’s tree represents 100%. The reverse does not make any sense and saying that last year's tree was 80% of this year's height presents problems. Teachers might address by using a ten foot tree that grew 20%. Determining the growth of two feet is easily done and of course twelve feet is 20% more than ten feet. One the "one" is established its height goes in the denominator. The height of the tree this year is unknown so it is the variable X but we do know that the height of the tree this year represents 120% of the height of last year.

This problem can also be stated as last year's tree was measured at 35 feet and this represents 80% of the height of this year's tree. The problem here seems similar to the previous one with a huge difference. The "one" is now this year's tree. Since we don't know the height of this year's tree then the variable X now takes its place in the denominator and the 35 is the numerator. The percentage that this division statement represents is equivalent to 80%. The percentage is always a reflection of the value in the numerator.

Percent Increase or Decrease

These problems reflect the percent something grows or shrinks. We can use the tree problem again here to show different perspectives. If a tree last year was 35 feet tall and this year it is 42 feet tall what percent did the tree increase? In percent increase problems determining the one is easy! It is the original number or where we started. Since we started with last year's tree and we want to know the growth the 35 is placed in the denominator and the 42 is placed in the numerator. When these are divided we get 1.25 which is 125%. Students must be aware that this does not conclude solving the problem. We now need to compare percents. If this year's tree is 120% of last year's tree and last year's tree is the "one" or 100% then the difference is 20% growth.

Another method is to divide the increase or decrease by the original amount. I have developed an acronym based on a local grocery store's name to help students solve the problem if they forget the formula. This of course is a common method used by many teachers to help students. I think these "tricks" have their place but they should come with some in depth discussion about why they are what they are. In this case the word is Bi-Lo. I tell the students to break it down into **Bi**g-**L**ittle divided by **o**riginal. The answer will be in decimal form and must then be converted into a percentage. This method works because when finding the difference between the numbers the student is finding the increase or decrease and of course then dividing it by the one which is the original number.

Money problems involving percent increase and decrease are very popular with students. The best example is to start with sale items. The quickest and most efficient way I have found to solve sale problems is to not determine the amount off and subtract it from the original price but instead to determine the percent that would be paid for the item. An example of this might be if a video game for $25.00 is on sale for 20% off then in actuality the buyer is only paying 80% of the price. Simply determine what number is

80% of $25.00 and this is what is paid. The same logic holds true for adding on sales tax or gratuity. The buyer has to pay the 100% price of the bill plus the tax or gratuity depending on the problem. If the video game had a sales tax of 8% then what percent is actually being paid at the counter? The answer is 108%. The students are left to determine what number is 108% of 25.

Activities

Lesson One “What is a Fraction?”
Objective: Students will learn all the elements of a fraction
Link to prior learning:

- Review students’ prior understanding of fractions and how to work with fractions utilizing all four basic operations.

Guided Practice:

- Students will be shown a pizza cut into eight slices. The teacher should discuss the value of the various slices and identify each slice with a fractional value.
- The teacher should discuss fractional values that total one, are greater than one and less than one.
- The teacher should show bars and utilizing the Singapore Model Method demonstrate how to add and subtract fractions with the bars. The teacher should lead to the discovery of why it is necessary to have a common denominator when adding and subtracting fractions. The discovery is based on size and how the fractional representations have to be compared to the same one (whole) in order to be added or subtracted.

Independent Practice:

- Students should be given practice problems to work on.

Lesson Two “Exploding Dots”
Objective: Students will be able to recognize place value of the base ten number system by utilizing exploding dots.
Link to prior learning:

- Use simple addition, subtraction, multiplication and division problems and have students determine the answers through whatever means they wish. When reviewing the answers have students explain how they arrived at their solutions i.e. by carrying, borrowing, etc.

Guided Practice:

- Utilizing a dry erase board (individual) students should be led through example problems of exploding dot machines such as binary and other base number systems and the teacher should have them display their answers on their boards to demonstrate understanding of the concept.

Independent Practice:

- Students will be given a handout with boxes and machine descriptions. They will complete the problems assigned.
- When reviewing the worksheet the teacher should bring everything together by discussing the base 10 machine and how it would be represented.

Lesson Three "Where in the World is Ciento"
Objective: Students will become familiar with decimal values and decimal representation and their relationship to percents.
Link to prior learning:

- Students should be made aware of the relationship of place values and decimal representation of a number.

Guided Practice:

- Define percent as a number divided by one hundred. The ratio of a number to one hundred.
  - Example: 45/100 is 45% 100/100 is 100% and coincidently it is also one.
- The fractions are turned into decimals and it is from these decimals that the percents are determined.
- The teacher should focus on the hundredth place value.
  - Example: 45/100 is .45 and since 5 is in the hundredths spot then this rather conveniently becomes 45%.
  - Example: 1/8 presents a problem because 8 does not go into 100 evenly so the percent is determined from the quotient of 1/8 which is .125. Since 2 is in the hundredth spot the percent is 12.5%. The teacher should explain that by definition percent is the ratio to one hundred and in this case this would be 12.5/100. Show students that 8(12.5) equals one hundred. 1/8 is equivalent to 12.5/100 that is all that is occurring.

Independent Practice:

- Students will practice changing fractions to decimals and then expressing the decimals as percents.
- Students may decide that it is easier to simply change the denominator to one hundred and then change the numerator proportionally.

Lesson Four "We found Ciento por Ciento" (A Hundred Percent)
Objective: Students will solve problems involving percents
Link to prior learning:

- Students will practice determining what percent of a number another number is based on the ratio.

Guided Practice:

- Students will learn how to compare quantities and determine percentages.

- Example: If bar A is 4 and bar B is 5 then bar A is 80% of bar B because of the resulting fraction 4/5. If bar B is compared to bar A then bar B is 125% of bar A because the fraction is now 5/4.

- Students will solve problems when a percent and quantity is known.
  - Example: Parcel A weighs 5 lbs and parcel B is 15% lighter than parcel A. What is the mass of parcel B?
    - Since 5kg is our one (100%) then parcel B is compared to it. 5/100 is .05 kg per unit and if we have 85 units multiply .05 by 85 and the answer is 4.25 kg.
    - Another way of looking at it is t set it up as a proportion since we are comparing A to B then it is A/B as a fraction which is equivalent to 85/100 so A/5=85/100 and solve this way.

Independent Practice:

- Students should practice solving problems using whatever method is better for them.

Bibliography

1.Hong, Kho Tek and Mei, Yeo Shu and Lim, James: *The Singapore Model Method for Learning Mathematics.* Singapore: EPB Pan Pacific, 2009.

2.Howe, Roger and Epp, Susanna. *Taking Place Value Seriously: Arithmetic, Estimation and Algebra.* http://www.maa.org/PMET/resources/PVHoweEpp-Nov2008.pdf

3.Liping, Ma: *Knwoing and Teaching Elementary Mathematics.* Mahwah, NJ: Lawrence Erlbaum Associates, 1999.