Musical and Exponential Models

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This curriculum unit is recommended for:
8th grade Math / Math I

Keywords: Major Scale, rhythm, time, guitar's strings, measure, pattern, base, exponent, function, exponential model

Teaching Standards: See Appendix 1 for teaching standards addressed in this unit.

Synopsis: This curriculum unit has been made to introduce the concept of exponential models, but also to relate mathematics with real world, to connect math concepts with contexts we don't usually explore. We don't need to be a music expert to use this unit. We just need to enjoy music, and be prepared to learn a little bit more about basic music concepts which will help us to understand the idea of exponential growing or decreasing. We don't notice, but exponential forms are around us, different cases of animal reproduction, spreading news, and some companies have a pyramidal structure with one headmaster and two or three people working directly with him or her, then every one of them is in charge of other two or three people, etc. The students will learn about how to identify an exponential model, how it works and apply this concept in real world situations. The students will also use tables and graphs to represent mathematically this situation and any other, in order to create an equation to solve the specific situation and others related. This unit includes activities with guitars, but it'd be possible with different strings instruments (violin, banjo, and ukulele). There is also an activity using graphic calculator TI 83Plus, but it's optional for the purpose of the unit.

I plan to teach this unit during the coming year in to 20 students from Math I and 48 students from 8th Grade Math in Collinswood Language Academy.

Musical And Exponential Models

Gilberto Franco

Introduction

Music and math are universal languages, like English or Spanish. You can learn them, and as you could become an English or Spanish speaker, you could also become a math or music speaker. You cannot memorize too many concepts about math but the most important skill is not about how much you can memorize, it is about how you can relate math concepts to real life. In other words, how can you use math in your life?

You don't need a complex context to find math. Daily events, routines, activities often involve a lot of math concepts. Just find them and enjoy the mathematics that are involved in doing them.

School Background

Collinswood Language Academy is a K-8 Spanish Immersion program with 750 students from different Latin American countries and the U.S. The school provides dual language instruction from Kindergarten to 8th grade. Kindergarten receives 85% of the daily instruction in Spanish, and 1st through 8th receives 50% of the instruction in English and 50% in Spanish (social studies and math are in Spanish).

One of the most challenging topics for the students is the instructional time for math in Spanish, because they have to relate their knowledge of their second language to their understanding in mathematics.

The 20 students that I will be working with are my honors Math I class. The class is composed of 50% Latin American students and 50% American students. No matter what their native language is, the students have good basic skills in Spanish as well as many of them have special interest in music (some of them play instruments and/or have been part of the school band).

Teacher Strategies

The students I'll be working with are curious, like to learn to ask questions to go beyond the class content so, planning classes for them means the teacher has to provide them with challenging questions, engaged activities to ensure they are focused in the class topic and enjoy learning. To work in this class unit I'll be using different PEAK[1]

[1] https://www.peaklearningsystems.com/resources/t4e/articles/

strategies which allow me and the students to follow an engaging path to understand and apply successfully the concepts about exponential models.

1. The first strategy I'll be using is Pre-Instruction Unit Planner. This strategy permits to introduce the students to a specific topic some classes before the topic will be developed in class it's divided into three different moments: Cloud, Sprinkle and Shower. The first moment (Cloud) shows just an idea about the topic we want to do. This could happen two or three weeks before the main class. The second moment (Sprinkle) show more information and give the students some ideas about the main topic. The third moment (Shower) is the closest before the class. The students will receive more information in order to generate a big questions which will guide them to the main topic of the unit class.

   The pre-instruction unit planner will start three weeks before when I'll introduce (or review) basic ideas about the major Scale (Cloud). In the middle of the week we'll take about 20 minutes to talk about *the major scale*. I'll show the structure of the major scale, whole notes flats and sharps.

   Two weeks before (Sprinkle) we'll be talking about how we can see the major scale in different instruments like flute, piano and guitar. This part could be done in 30 minutes of one class or 15 minutes for two classes.

   Finally, the week before the big event (Shower), we'll talk about the sounds, how to produce it and their duration. At the end of this part, they will receive a big question which is the introduction to the topic of the unit: How to relate all of these ideas with the concept of exponential models?

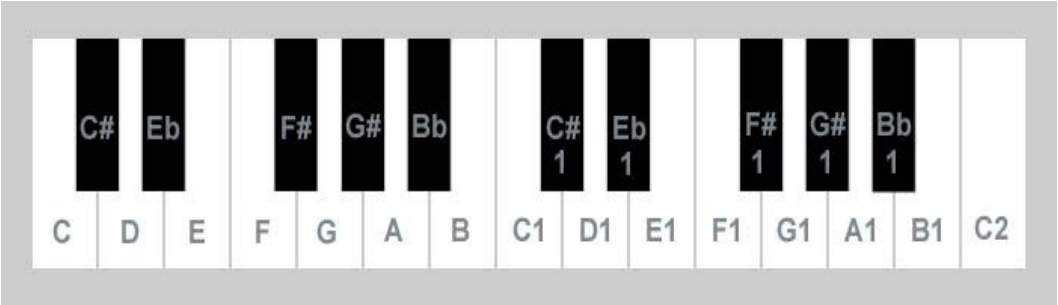
2. The agenda for the day is a good way to start because they know what is going to happen during the class and also, they can control their own time for every activity. This agenda could contain aspects like checking homework, exploration time, explanation, exercises, discussion time, etc. According to the unit class, during the three weeks before the main class, we'll include the topics explained before (cloud, sprinkle and shower times).

3. Turn and talk is one of the best strategies I like to use because some students don't like to talk in front of the whole class but they feel more confident talking with just one or two students. After the one minute for turn and talk, they can share what they'd talked about. This strategy is perfect to analyze the results of the tables they have to complete because some of them could identify different things than expected or, if they didn't find anything the teacher could guide them with different questions like: are the numbers increasing or decreasing? Is it a linear pattern? Look at the same sounds' name and see how they are related.

4. Content Graffiti is a good way to summarize the big ideas they are collecting during the class. This activity is a good strategy for working in groups because every student can share their own ideas and complement others. The graffiti could be written in a piece of paper or in one of the desks. At the end of the class we'll share the information in every content graffiti to conclude a class session.

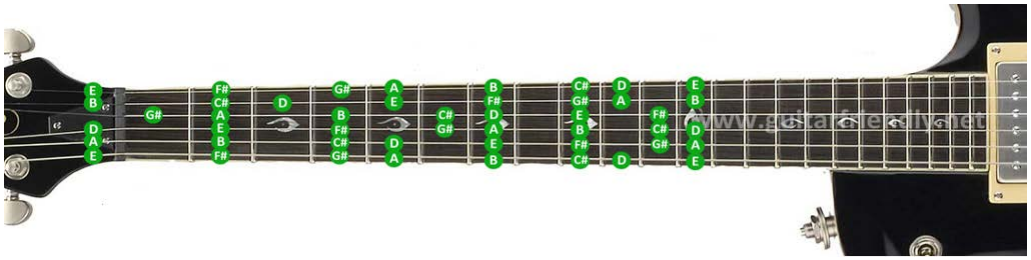
Activities

Now let's talk about our main topic that is the relationship between math and music. My class unit will start talking about the major scale, its structure and how it works. I'll show this scale using a piano:



Picture 1

And a guitar.



Picture 2

After this introduction we'll break into groups of two. Each group will take a guitar and a measuring tape. They are going to take the measure of the first chord of the guitar with the measuring tape, this is the note E. Then they'll measure each note and fill a table with this information. (I'll include an example about the possible results. Data may be different according to the guitar they use, some of them are bigger than others).

NOTE	MEASURE (cm)		NOTE	MEASURE (cm)
E	63.3		E_1	31.65
F	59.8		F_1	29.9
F#	56.4		$F\#_1$	28.2
G	53.3		G_1	26.15
G#	50.3		$G\#_1$	25.15
A	47.5		A_1	23.75
A#	44.8		$A\#_1$	22.4
B	42.3		B_1	21.15
C	39.9		C_1	19.95
C#	37.7		$C\#_1$	18.85
D	35.6		D_1	17.8
D#	33.6		$D\#_1$	16.8

Table 1

After the students are done with the table, we'll analyze the results and find something that they can capture their attention. Depending on the group, the teacher must guide the students with some questions in order to discover what topic they are looking for: when they start the second scale, the measure is exactly half of the measure of the same note in the first scale.
The questions could be:

- Why does the sound change every time we move the finger over the chord?
- What do you find when you take a look of the whole table and then compare the same notes?
- What could be the measure of the note E_2?

After they find out the relationship between the measures of the same notes, we'll make a second table but using only one specific note. (E.g. a table with the measures of the note E)

		Note	Measure (cm)
Initial value	0	E	63.3
End of 1st scale	1	E_1	31.7
End of 2nd scale	2	E_2	15.9
End of 3rd scale	3	E_3	7.95

Table 2

Now, let's do the math that relates to the music! The teacher will give the following question to the students: How can we operate the initial value to get the next one?

After a few time of turn and talk, they must find that the operation must be a multiplication by ½. So we'll make one more column in the table and rewrite the measurements in terms of the product of the first term and ½:

		Note	Measure (cm)	
Initial value	0	E	63.3	63.3
End of 1st scale	1	E_1	31.65	63.3 x ½
End of 2nd scale	2	E_2	15.875	63.3 x ½ x ½
End of 3rd scale	3	E_3	7.95	63.3 x ½ x ½ x ½

Table 3

And then we'll add another column to express these products as a power (I'd recommend that the last column could be filled from the bottom to the top, because this way will be helpful to explain why a number to the power of ZERO is equal to 1, in this case, why ½ to the zero power is equal to one):

		Note	Measure (cm)	as a product	as a power
Initial value	0	E	63.3	63.3	63.3
End of 1st scale	1	E_1	31.65	63.3 x ½	$63.3\ x\ \left(\frac{1}{2}\right)^1$
End of 2nd scale	2	E_2	15.875	63.3 x ½ x ½	$63.3\ x\ \left(\frac{1}{2}\right)^2$
End of 3rd scale	3	E_3	7.95	63.3 x ½ x ½ x ½	$63.3\ x\ \left(\frac{1}{2}\right)^3$

Table 4

The last part of the class is about the concept of exponential model.
We'll write the concept according to the work we've done during the class. To do this we'll answer the following questions:

- How many variables we've been relating?
- We have the initial value and we've multiplied by a number. What could be the number for this case?
- What is the equation for an exponential model?

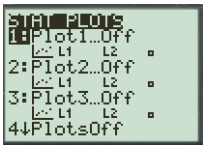
Now we'll define the Exponential model. We can turn and talk again to create the definition and then share with the rest of the class. After that, the teacher will give the definition to unify the ideas:

An Exponential model is a relationship between two variables, which can be modeled by the function $y = ab^x$, where a and b are greater than 0 and b≠1.

Another way to understand this is by graphing, so, we'll graph the *table 2* using the TI-83 PLUS calculator.

The first step we need to do before graphing is setting the calculator.

Press 2nd and y= to go to the stat plot settings

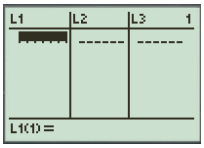


Go to plot 1 and press enter

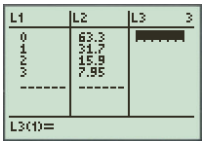
And then enter again

Press CLEAR and then STAT

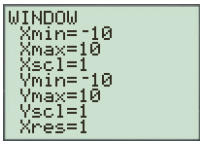
And ENTER



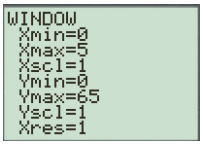
We'll write the first column in L1 and the second in L2



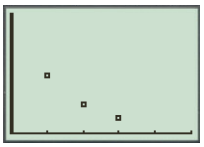
Now we'll go to WINDOW



And we'll change the values according to the table



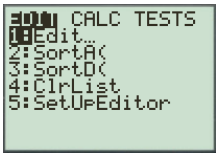
And then press GRAPH



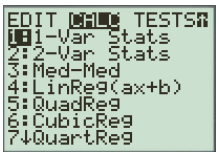
We can see only three points because the first one is over the Y-axis.

Now we can find the equation of the exponential model with the following instructions:

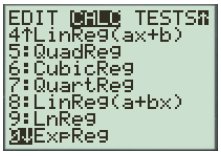
Press STAT



Go to CALC

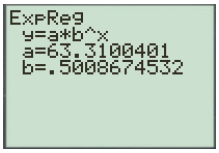


Go down to number ZERO



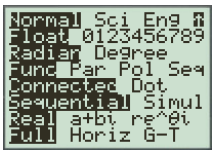
Press ENTER

And again ENTER

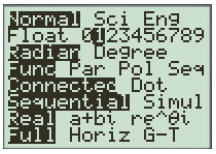


Due to the table has values with only one decimal number, we can round these values to one decimal number too.

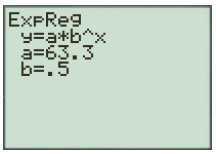
Press MODE



Go to FLOAT and choose the number 1



Press CLEAR and then ENTER



So, the equation will be: $y = 63.3 \times 0.5^x$

As you can see, the value 63.3 is the first value of the table and 0.5 means that every value will be half of the value before. As a homework, the students must bring to the next class examples of real world situations involving exponential models.

One additional activity about the exponential model is about comparing different context about exponential models and power and exponents models to learn the differences in terms of structure of the model and identifying the word that could guide us to make the exponential or the power and exponent models. A possible warming up for this class could be:

Which of the following operations is growing faster? Explain your reasons

A	B	C	D
2+5	2x5	2^5	5^2
3+5	3x5	3^5	5^3

The students can turn and talk and then share with the group their reasons about the question.

More about math and music

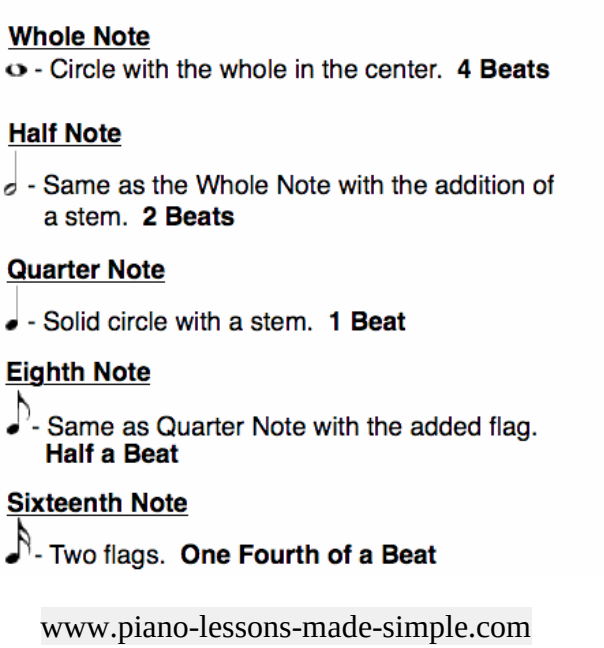
"*The founder of what is now considered the standard music stave was Guido d'Arezzo,* [9] *an Italian Benedictine monk who lived from about 991 until after 1033. He taught the use of solmization syllables based on a hymn to Saint John the Baptist, which begins Ut Queant Laxis and was written by the Lombard historian Paul the Deacon. The first stanza is:*

1. *Ut queant laxis*
2. *resonare fibris,*
3. *Mira gestorum*
4. *famuli tuorum,*
5. *Solve polluti*
6. *labii reatum,*
7. *Sancte Iohannes.*

Guido used the first syllable of each line, Ut, Re, Mi, Fa, Sol, and La, to read notated music in terms of hexachords; they were not note names, and each could, depending on context, be applied to any note. In the 17th century, Ut was changed in most countries except France to the easily singable, "open" syllable Do, said to have been taken from the name of the Italian theorist Giovanni Battista Doni".[2]
The information above refers to the names for the musical notes. Those are the main codes to write and play music. There are a lot of more terms and notation, however we'll just need a few of them.

[2] https://en.wikipedia.org/wiki/Musical_notation

A sheet music is the written way to understand the music. Every sound, every silent can be written using a staff and musical symbols which show the difference between the sounds. The following chart shows the differences between the sounds in terms of time duration:



Notice that the names of the musical notes use fractions.

Rhythm

Rhythm, in music, the placement of sounds in time. In its most general sense rhythm (Greek rhythmos, derived from rhein, "to flow") is an ordered alternation of contrasting elements. The notion of rhythm also occurs in other arts (e.g., poetry, painting, sculpture, and architecture) as well as in nature (e.g., biological rhythms).

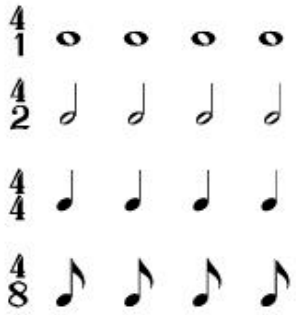
Rhythm is music's pattern in time. Whatever other elements a given piece of music may have (e.g., patterns in pitch or timbre), rhythm is the one indispensable element of all music. Rhythm can exist without melody, as in the drumbeats of primitive music, but melody cannot exist without rhythm. In music that has both harmony and melody, the rhythmic structure cannot be separated from them. Plato's observation that rhythm is "an order of movement" provides a convenient analytical starting point.[3]

[3] *http://www.britannica.com/art/rhythm-music*

Time

The mind apparently seeks some organizing principle in the perception of music, and if a grouping of sounds is not objectively present it imposes one of its own. Experiments show that the mind instinctively groups regular and identical sounds into twos and threes, stressing every second or third beat, and thus creates from an otherwise monotonous series a succession of strong and weak beats.

In music such grouping is achieved by actual stress; i.e., by periodically making one note stronger than the others. When the stress occurs at regular intervals, the beats fall into natural time measures. Although in European music the concept of time measures reaches back to a remote age, only since the 15th century have they been indicated by means of bar lines. Thus, the terms measure and bar are often used interchangeably. The time measure is indicated at the opening of a piece by a time signature; e.g., $^{2}/_{4}$, $^{4}/_{8}$, $^{3}/_{4}$, $^{6}/_{8}$. The length of each beat in a measure may be a time unit of short or long duration:



The signature $^{4}/_{1}$ (above) means that the whole note (1) is the unit in each measure, and there are four (4) of them to each measure. In the second illustration, $^{4}/_{2}$, the half note (2) is the unit of measurement, with four of them (4) to each measure, etc.

The two basic types of time measure have either two or three beats and admit of many different notations.[4]

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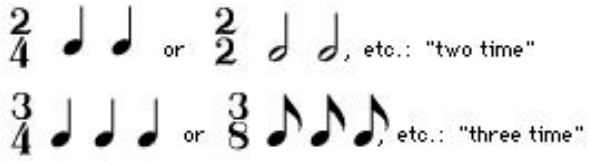
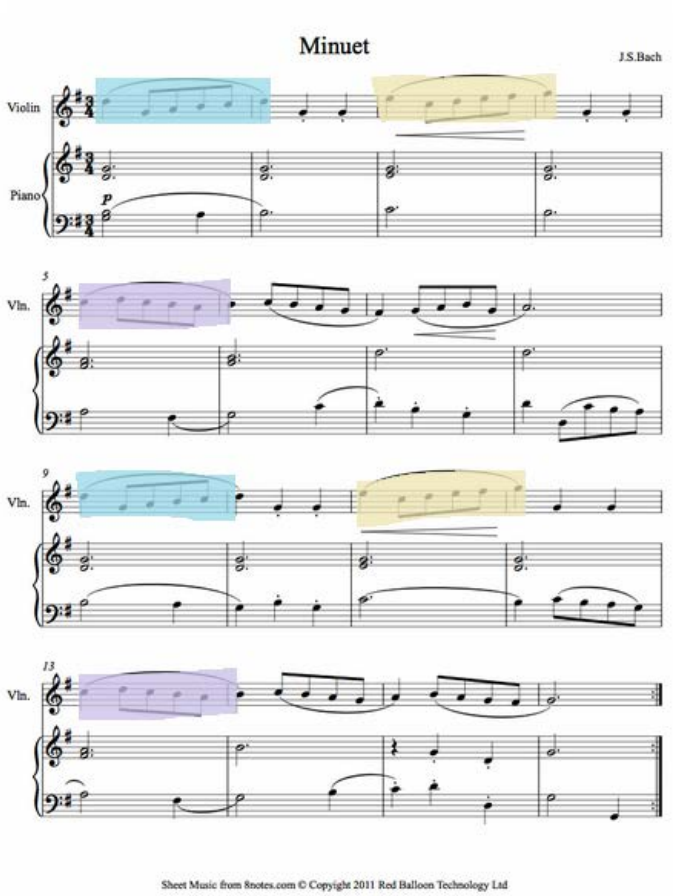
[4] http://www.britannica.com/art/rhythm-music



We can find a lot of examples around us about this amazing relationship, for example the Johann Sebastian Bach's music is full of numbers and patterns (http://pages.jh.edu/~jhumag/0298web/math.html) and his music has been studied a lot of times (https://www.etbu.edu/files/2113/8608/9827/Jennifer_Shafer.pdf). But we'll talk about two simple relationships between math and music. One of them is fractions and the other one is patterns.

1. When we talk about patterns in math we relate repetitive series of numbers, actions, shapes, and this is what music is. Let's see an example:



We can identify some sets of notes which are repeated during the same melody, but also there are a *second pattern* with them, notice the order of those patterns: light blue, yellow and lavender, and light blue, yellow and lavender again.

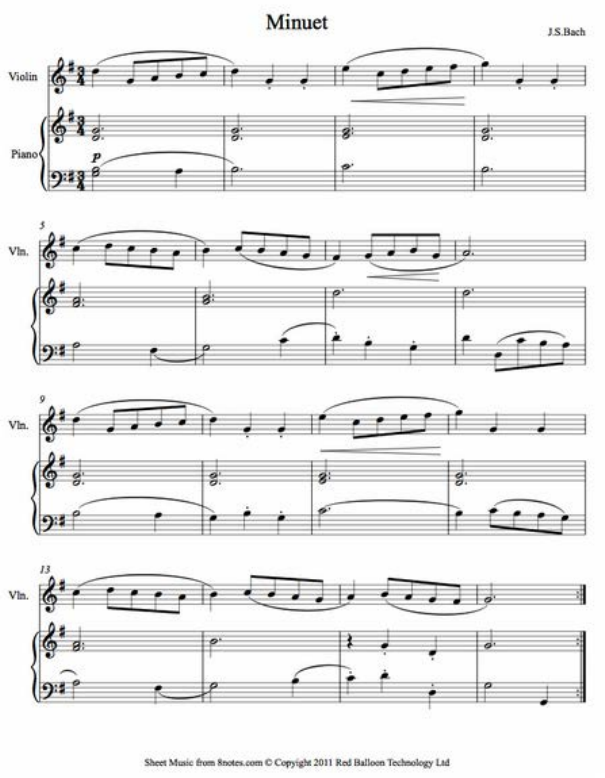
Now, you can create your own patterns using music. Clapping our hands three times, wait a second and then clap your hands three times again and again and again. This is a simple example but we can create more challenged examples like 2 or 3 people every one with a different pattern of clapping:

Person 1: x x x o o o x x x o o o x x x o o o x x x o o o x x x o o o x x x o o o x
Person 2: x o o x o o x o o x o o x o o x o o x o o x o o x o o x o o x o o x o o x
Person 3: x o o o x o x o o o x o x o o o x o x o o o x o x o o o x o x o o o x o x

Where *x* means clap and *o* represents silence. Try it!!!

2. Now we'll talk about fractions.

Let's see the music sheet again:



As I explained before, the sheet's tempo is ¾. It means that every set of notes, separated by a line must measure ¾. Let's take the first set of notes:

There are 5 sounds: 1 quarter note and 4 eighth notes. According to their measure, we'll add their values:

$$\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}$$

$$=\frac{2}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}$$

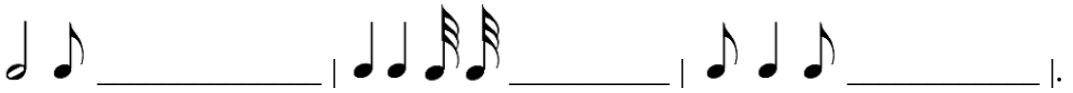
$$=\frac{6}{8}$$

$$=\frac{3}{4}$$

And it works for every set of notes. The second set has 3 notes, each one of them measures ¼, so the total measure is ¾.

According to this, one activity about math and fractions could be:

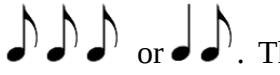
Complete each tempo and then express each one as a sum of fractions.

4/4  |.

The number on the left means that the sum of the notes must be 1. So, in the first tempo we have 2 and ½ so:

$$2+\frac{1}{2}=\frac{5}{2}$$

It means we need $\frac{3}{2}$ more to complete this tempo.

One option could be  or . The same reasoning for the second and third tempos.

Music and math are languages like English or Spanish. All of them have some codes and we can relate those codes to create everything inside each language. For example: English has 27 different codes and every combination of 1, 2 or more codes is called word, and the combination of words is called sentence.

In music, there are 12 codes called notes, the combination of notes are tempos and the combination of tempos is called melody. Math has the same order. There are 10 codes (0,1,2,3,4,5,6,7,8,9), the combination of 1, 2 or more codes are called numbers, the combination of numbers is a polynomial.
As we can see, every language is the same structure but different codes and names. If we are able to learn our native language, we can learn everything we want. Now, our language proficiency depends on how to manage the basic concepts, rules and structures. If you have a good vocabulary and your grammar level is good, with practice you can understand everything you read in your Language, so if you learn a second language, it must be the same. So, learning math is not as different as learning any language. Try it! Learn it! Enjoy it!

Additional resources

I will include a video to explain how the sound works, through guitar string waves.

https://www.youtube.com/watch?v=RUpjYDteYcg

STUDENT'S MATERIAL

Student's name: _____ Date: _____

Complete the following tables according to the class instructions.
Table 1

NOTE	MEASURE (cm)		NOTE	MEASURE (cm)
E			E_1	
F			F_1	
F#			$F\#_1$	
G			G_1	
G#			$G\#_1$	
A			A_1	
A#			$A\#_1$	
B			B_1	
C			C_1	
C#			$C\#_1$	
D			D_1	
D#			$D\#_1$	

Related questions:

- Why does the sound change every time we move the finger over the cord?
  _____
  _____
- What do you find when you take a look of the whole table and then compare the same notes?
  _____
  _____
- What could be the measure of the note E_2?
  _____

Table 2

		Note	Measure (cm)
Initial value	0	E	
End of 1st scale	1	E_1	
End of 2nd scale	2	E_2	
End of 3rd scale	3	E_3	

Related Question:

- How can we operate the initial value to get the next one?

Table 3

		Note	Measure (cm)	
Initial value	0	E		
End of 1st scale	1	E_1		
End of 2nd scale	2	E_2		
End of 3rd scale	3	E_3		

Table 4

		Note	Measure (cm)	as a product	as a power
Initial value	0	E			
End of 1st scale	1	E_1			
End of 2nd scale	2	E_2			
End of 3rd scale	3	E_3			

Related questions

- Which variables have we been relating?
  _____
- We have the initial value and we've multiplied by a number. What could be the number for this case?
  _____
- What is the equation for an exponential model?
  _____
- What is the equation for this exponential model?
  _____
- What is the definition of Exponential model? (Write your definition according to what we've discovered during this unit, then, share with your partner)
  _____
  _____
  _____
  _____

Which of the following operations is growing faster? Explain your reasons

A	B	C	D
2+5	2x5	2^5	5^2
3+5	3x5	3^5	5^3

_____
_____
_____
_____

Complete each tempo and then express each one as a sum of fractions.

4/4  _____ | _____ | _____ |.

Appendix 1: Implementing Teaching Standards

CCSS.MATH.CONTENT.8.F.A.1
Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output.[1]

CCSS.MATH.CONTENT.8.F.B.5
Describe qualitatively the functional relationship between two quantities by analyzing a graph (e.g., where the function is increasing or decreasing, linear or nonlinear). Sketch a graph that exhibits the qualitative features of a function that has been described verbally.

CCSS.MATH.CONTENT.HSF.IF.A.1
Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$.

CCSS.MATH.CONTENT.HSF.IF.C.7
Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.*

CCSS.MATH.CONTENT.HSF.IF.C.7.E
Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude.

CCSS.MATH.CONTENT.HSF.LE.B.5
Interpret the parameters in a linear or exponential function in terms of a context.

Resources

1. Exponents and Exponential Functions: Algebra Nspirations (This resource could be used if you have a UNC library account) This video includes basic information about exponential functions. Dr. Monica Neagoy gives a great compilation of ideas about exponential functions: fundamental properties and real world situations. https://librarylink.uncc.edu/login?url=http://digital.films.com/PortalPlaylists.aspx?aid=29420&xtid=53218

2. The following videos contain basic ideas about exponential functions. You can understand how to write and graph exponential functions and differences between exponential and polynomial functions.

   Introduction to Exponential functions:
   https://www.khanacademy.org/math/algebra/introduction-to-exponential-functions/exponential-growth-and-decay/v/exponential-growth-functions

   Graphing exponential functions given the formula:
   https://www.khanacademy.org/math/algebra/introduction-to-exponential-functions/graphs-of-basic-exponential-functions/v/graphing-exponential-functions

   How to distinguish between linear and exponential functions:
   https://www.khanacademy.org/math/algebra/introduction-to-exponential-functions/comparing-exponential-and-polynomial-functions/e/understanding-linear-and-exponential-models

3. http://www.intmath.com/exponential-logarithmic-functions/exponential-log-functions-intro.php This website contains basic information about exponential functions: definition, algebraic and graphic structure.

4. Lappan, Glenda. *Connected Mathematics 2*. Boston, Mass.: Pearson, 2009.
   This text book contains a unit about exponential functions called *Growing, growing and growing*. It's correlated with the Common Core Standards.

5. http://www.corestandards.org/Math/Practice/. This site is for teachers to access the Common Core Process Standards and Objective Standards for Mathematics.

6. https://www.peaklearningsystems.com/resources/t4e/articles/ This site can be used for teachers to read and learn about PEAK (Performance Excellence for All Kids) classroom strategies

7. https://en.wikipedia.org/wiki/Musical_notation Teachers and students can use this link to read about musical notation, how to differentiate tones and rhythms.

8. http://www.britannica.com/art/rhythm-music. Another link to read about music concepts. You don't need to be a musician to access this information

Classroom materials

1. Students' worksheet
2. Measuring tape
3. Guitars
4. TI-83 PLUS (optional) this is for an additional activity but it's not essential for the unit.

Bibliography

1. https://www.peaklearningsystems.com/resources/t4e/articles/

2. https://en.wikipedia.org/wiki/Musical_notation

3. www.piano-lessons-made-simple.com

4. http://www.britannica.com/art/rhythm-music