‘How You Do’: The How to Do It

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This curriculum unit is recommended for:
(Middle and High School Math and Science

Keywords Binomial coefficients, combination, permutation, tree diagrams, Pascal triangle, pair share, ZomeTools, triangular numbers

Teaching Standards: See Appendix 1 for teaching standards addressed in this unit

Synopsis: Hand shake problems or the combination of things taken 2 at a time are found in many areas of mathematics science and life. This would include setting up tournaments, where everyone plays each other once, and pairings. The numbers are part of a binomial expansion $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$ it represent 2 of terms when n>2 and a diagonal for Pascal Triangle...
These lessons start with the idea of nine Supreme Court Justices, who shake hands at the beginning of each session, which is a custom almost 100 years old; and the question how many handshakes does it take? This eventually leads in the lesson to ‘How many combinations of ‘n’ things taken 2 at a time, The students discover the answers to several questions involving this, different methods of finding an answer and eventually that nCr is equal to (n(n-1))/2. The nine justices do a total of 36 handshakes.

I plan to teach this unit during the coming year in to (**15**) students in (**a variety of math and science courses**).

Teaching Standards

MATHEMATICS: Standards for Mathematical Practice and Common Core

1. Make sense of problems and persevere in solving them.

2. Reason abstractly and quantitatively

3. Construct viable arguments and critique the reasoning of others.

4. Model with mathematics.

5. Use appropriate tools strategically.

7. Look for and make use of structure.

8. Look for and express regularity in repeated reasoning

How Do You Do

Introduction/Rationale: My grandfather, when around an infant or small toddler, would say/ask ‘How ya do’. I want my students to learn and be introduced to problem solving, so ‘How Do You Do’ is the introduction and methods for what I hope is a rewarding journey.

I find ‘Handshake Problems’ and the solution has application for many problems. When my children were young, they sat and participated in a summer class middle school math teacher class at Baldwin Wallace University in Berea Ohio with me. They talked math all the way home (we had homework), at the supper table and again in the morning as we drove to school (75 mile trip each way). This was amazing to me because the oldest had difficulty with math.

Objectives: I plan to address all the Standards for Mathematical Practice in the activities and lessons. Not all lessons will be applicable to all grade or ability levels. I think the ability to see and/or find a pattern can work for all my students.

Demographics: I teach a diverse group of students that can change from week to week, except for one common issue. My students are in treatment for alcohol and/or drug abuse. Our students can be in grades 6 to 12 and under 18 years of age. Often they are behind in classes or may not have been in school for a while, as much as 2 years. Most are males and are in living in a residential setting (in the building). They come from all over the state not just CMS and have different family issues, incomes, living arrangements and other background issues. Unlike in a regular school, I know what substances they abuse and their levels of use. They are monitored, tested, weekly, physically checked (patted down) and ‘wanded’ often. They have a substance counselor, often PO’S and other court appointed people. My students are often minorities and from poverty but that changes all the time. As of now we have 7 students; 3 Hispanic, 2 Caucasian, 2 African American (1 from Africa) all males.

I teach math and science in our program it is possible to have 8 students in a class and deal with 8 subjects in one hour of class.

Content Background: These lesson are written for a class greater than what I normally teach.

These lesson will explore a simple case of a combinatorial problem when choosing 2 things at a time. Students will learn a distinction between combinations and permutations problem. Students will explore various ways of solving problems and learn that there is usually more than one way to do this. They will be asked to keep a journal about what they saw, did, found out and how they felt about these activities. This will address a Social Studies and English standards also. Communication of ideas is a great concern, I know there are course and actors like Alan Alda whose mission is to help technical people communicate their ideas effectively.

Strategies and Activities

Supreme Court Problem: When the nine Justices shake hands with each other, how many handshakes does it take?

First (easiest) Students will act out, simulate, using those present and dolls for 2 to 15 people and record their results on the board.

As I prepare each lesson, I will assign it to a student who will present it like a famous mathematician from this list. Except the very first lesson we will use Chief Justice of the Supreme Court; Melville W Fuller and the somewhat famous handshake problem.

Bertrand Russell

Carl Friedrich Gauss

Leonhard Euler

Jacob Bernoulli

Blaise Pascal

Pierre de Fermat

René Descartes

Leonardo of Pisa (Fibonacci)

Archimedes

Days to follow: Prior to the start of each lesson

The assigned student for day one will use the following information (or others if they choose). A student will act like Justice Fuller (then Chief Justice of the US Supreme Court) began the practice or custom (still used today) that at the beginning or start of each day, to shake hands with every other justice. There are nine justices and the students will act this out (with or without robes). http://www.scholastic.com/teachers/article/court-and-its-traditions#top.

Day One: Simulate or Act it Out.

'Justice Fuller' will choose 8 other student (Justices A to H) and they will act as if this was a new day at court and shake hands with each other, no repeating of any pairs. It should go something like this: Justice Fuller will shake 8 hands, Justice A will shake 7 hands, Justice B will shake 6 hands and so on. The rest of the class should keep track of the total number (36).

Pair share (talk to your neighbor) what you saw, then enter into your Journals what you experienced and how you felt about it.

Day 2: Collect Data and Make a Table.

Introduction of the historical person for the day.

You have decided to start a new family tradition for a special winter holiday, like Christmas. Each child will get a unique bag of money and other goodies to give; and a large bag to put their gifts in that they will receive from each other as they greet each other and introduce themselves or wish a 'happy holiday'. We have bags of pennies, nickels, dimes, quarters, half dollars, silver dollars, paper dollars, some colored or flavored candy canes. There is only one bag of each thing to start. You will need to go to the store and bank to get the needed supplies. Each bag will have the same number of items in it. Why? If everyone greets everyone else, will they end up with the same number of each item? We want to be fair, because if you have the bag of pennies and someone else has the dollar to start, will everyone else's bags be fair and equal?

If there are 10 children and there will be nothing left in your original bag, how many things do you need to have to start with in each bag? If you have dollars and give them all away you will not have one for yourself. Remember you are fair but frugal, these things might not grow on trees. The questions are, how many things in each bag? How many 'transaction' will occur?

We are going to first solve a simpler problem by using plastic fruit and plastic coins. Break into groups of 4; remember you give something and get something back. Act it out; make a table; and report your results in 20 minutes. Class will discuss what they did and found out.

Pair share (talk to your neighbor) what you saw, then enter into your Journals what you experienced and how you felt about it.

Day 3: Find a Pattern

Introduction of the historical person for the day. Today students will collect data and make a table. Suppose we have 26 people in a room and everyone shakes hands with everyone else, once and we are going to record this on the board. Person ‘A’ shakes hands with how many people? 25 How many people does person ‘B’ has already shaken hands with ‘A” but has 24 people they haven’t; person ‘C’ has shaken hands with ‘ A’ and ‘B’ they will have 23 people to shake hands with now. ‘H’ the 8^{th} person has how many handshakes to make? 18 and person ‘z’ how many? 0. I would write the 26 alphabet letters on the board and the numbers below them. Visit Day one and Day two goals and outcomes.

Day 3: Find the Pattern

Daily Goals

Create a chart using gathered data.

Students will see that there is a pattern and a shortcut when the gathered numbers are added.

Warm up Historical person of the day

Lesson “How Do You Do? How Do You Do?”

Prep: Create a two column table on the board. Letters of the alphabet will be written in one column. The other column will be partially filled out. (Students will complete the chart).

Introduction:

1. Revisit Day One and Day Two goals and outcomes.
2. Today you will continue to learn through what I have entitled “How Do You Do? How Do You Do?” That there is a pattern and a shortcut for counting handshake combinations.
3. We have 26 people in a room. Everyone will shake hands with everyone else in the room only one time. When Person A shakes hands with Person B; it is the same as B shaking hands with A. No need to repeat.

Activity THINK/PAIR/SHARE: How many people will Person A shake hands with? (25)

Tell students that there is one exception – What is it? (Person Z or 26 will not initiate a handshake, so the number for that line is ‘0’)

The students will fill in the column on the board with corresponding quantity of handshake combinations.

Wrap up

If I add a 27th person, How many handshakes do we add (26 for a total of 351, a triangular number)

Journal

NOTES Students will be led to understand that each letter of the alphabet combined with another letter will always equal the number 25.

Person B will have shaken hands with 24 people, Person C will have shaken hands with 23 people and so on.

Person H (the eighth person) will have shaken hands with 18 people. Person Z will have shaken hands with NO people.

A	(25)
B	(24)
C	(23)

X	(2)
Y	(1)
Z	(0)

Factors: A+Z=25; B+Y=25; C+X=25 and so on you will have 13 pairs of 25 or (26/2)*25 or 325 (a triangular number)

Day 4: Build a Model

Introduction of the historical person for the day. We are going to build several models today, you may do it any of these ways. Use a simple example for each using 4 things A B C and D. Tree diagram, person A would have 3 lines or limbs coming out, B would have 2 lines, C would have 1 line and D none (D would already shaken hands with everyone). We have 3+2+1=6

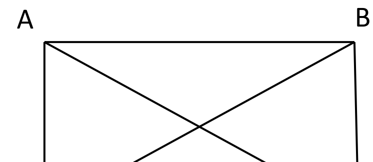
and a triangle shape diagram

AB AC AD

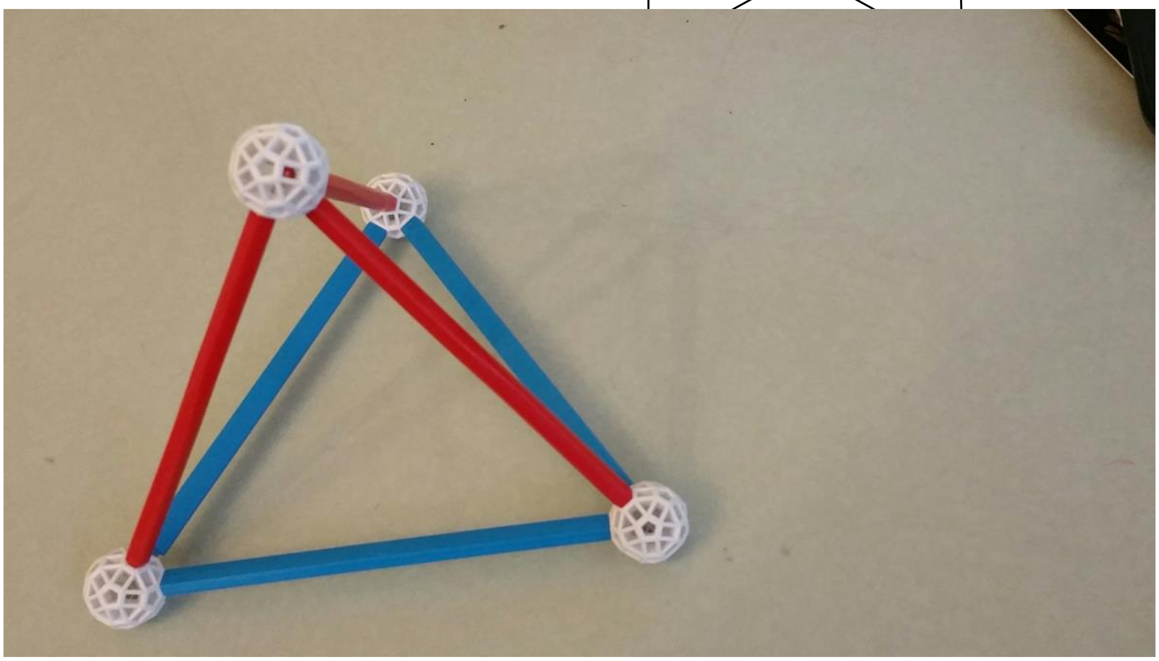
BC BD

CD

or in 2 dimensions



6 lines connecting the vertices.



Or by using ZomeTools I can in 3 dimensions, show 6 diagonals and 6 handshakes, 3 from each vertices.

When N=5, the number of handshakes or combinations is = 10 (4+3+2+1).

By using minions or any other figurines, I am showing when n=6 there are 5+4+3+2+1 = 15.

Students will now create one for 10 ‘people’ then 15. It takes practice and thinking ahead.

Pair share (talk to your neighbor) what you saw, then enter into your Journals what you experienced and how you felt about it.

Day 5 Graphing and Functions

Introduction of the historical person for the day. Define a function. Make a chart of 2 to 15 people shaking hands, then we are going to plot these points. Is this a function? Yes 1 input 1 output and it passes the vertical line test. Is it linear? No. How does this relate to combinations? What happens if I graph it and find equation of best fit? Quadratic

Let us talk about what we are seeing, if we have ‘n’ people. The nth person will shake hands with (n-1) persons; the ‘n-1’ person will shake hands with (n-2) people and so on sounds like a factorial except this is addition. But we are only going about half way and let’s think about the short cut we made ay taking a big number and a small number (equaled our ‘n’) adding them together we do that about (n-1)/2 times. When we do something called combination

$$\frac{n!}{r!(n-r)!} = \binom{n}{r}$$

where ***n*** is the number of things to choose
from, and we choose ***r*** of them
(No repetition, order doesn't matter)

For r=2, 2 people at a time we get the expression: n(n-1)/2. Let us get this from the above formula. r=2

N(N-1)(N-2)! DIVIDED BY (2 !(N-2)! Since 2 factorial is 2 we are left with n(n-1)/2 the (N-2)! Cancel out.

Pair share (talk to your neighbor) what you saw, then enter into your Journals what you experienced and how you felt about it.

Additional days or extensions for lessons depending on the grade and ability of the students can include.

Pascal Triangle can be used for answers if we go to the 4th row and look at the diagonal starting with 3 and go down along the side (3 places in) will give us the number of combinations taken 2 at a time (3,6,10,15,21…). We may have already learned how to use this tool for the coefficients in a binomial expansion.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 200 252 200 120 45 10 1

http://ohiorc.org/for/math/stella/

This a problem from Stella Stunners which is part of the above website .

At a party, everybody shakes hands with everybody else exactly once. A fly on the wall observes 36 handshakes total. How many people were at the party?

Solution

One approach to this problem would be to start with simple cases and look for a pattern:

2 people → 1 handshake

3 people → 3 handshakes

4 people → 6 handshakes, etc.

A more elegant approach uses some algebra. If there are n people, each person shakes hands n - 1 times. But $n(n - 1)$ counts every handshake twice (A with B and B with A is just a single handshake). So the total number of handshakes is $n(n - 1)/2$.

Setting $n(n - 1)/2 = 36$,

we get $n(n - 1) = 72$

so $n^2 - n = 72$

or $n^2 - n - 72 = 0$.

Factoring, $(n - 9)(n + 8) = 0$, and $n = 9$.

Additional or Extension Activities (could be done with ZomeTools**)**

Both of these activities are from Connecting Geometry to Advanced Placement Mathematics **(**http://mason.gmu.edu/~jsuh4/impact/Handshake_Problem%20teaching.pdf)

Finding a formula for the number of diagonals in a polygon.

A graphing calculator can be used for finding combinations and permutations. The NSPIRE series (including the CX (color) https://education.ti.com/en/us/activity/detail?id=CE05857DAFBE4447901DFD7373D58B25). (These allow the creation of documents (lessons) that the student can then work through**)**

Developing a formula for finding the number of diagonals in an n-sided polygon,

This is a small group or partner activity. Student should make a chart:

The Number of Sides	The Number of Diagonals
3	0
4	2
5	________(5)
6	________(9)
7	________(14)
8	________(20)
N	_________ ((N-1)(N-2)/2) -1

Show that (n)(n-3)/2 also works and are they the same? Set them equal to each other. (N^2-3N+2)/2 -1 = (N^2 -3N)/2

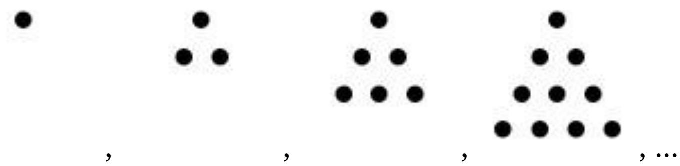
Multiply both sides by 2 (do not forget to multiply the -1 by 2) they are equal.

Diagonals do not include the corner and the corners adjacent to it, these form the edges (like a perimeter).

The Rock Splitting Problems (http://math2.uncc.edu/~hbreiter) **or Grapefruit Stacking Problem (http://ohiorc.org/for/math/stella/problems/problem.aspx?id=323#)**

On a special day at the West Side Market, the grapefruit are arranged in a compact stack of filled-in equilateral triangles with 14 grapefruit on each edge of the bottom triangle, 13 on each edge of the one above, and so on all the way up to the top where there is 1 graperfruit sitting all alone. How many grapefruit are in the entire stack?

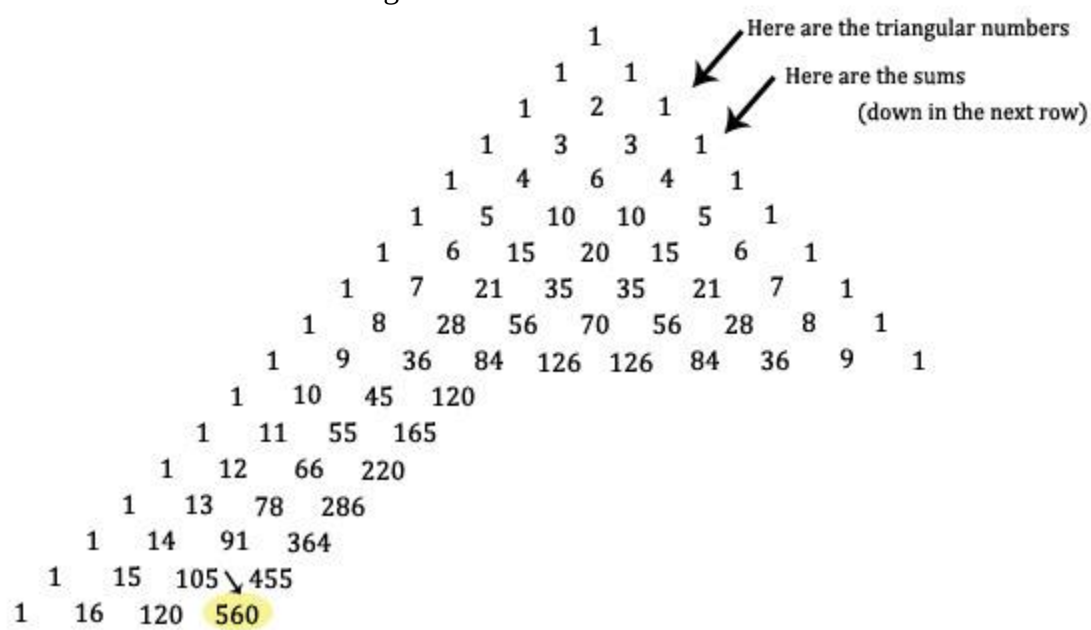
Starting at the top, the layers look like this:



The totals of the layers are 1, 1 + 2, 1 + 2 + 3, ..., 1 + 2 + ... + 14.

You can use the formula $1 + \ldots + n = (n(n + 1))/2$ repeatedly to find the total.

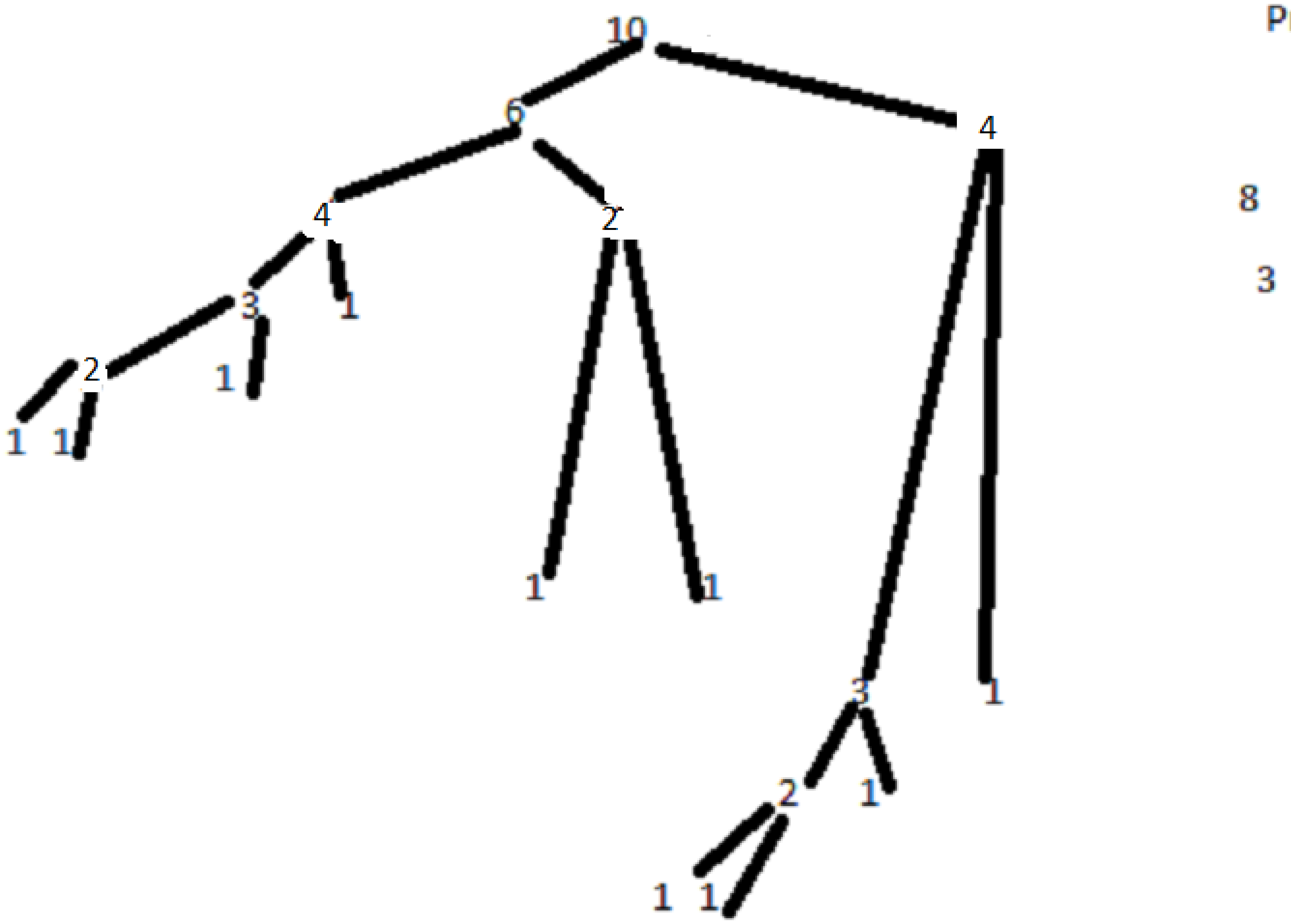
You can also use Pascal's triangle:



$1 + 2 + \ldots + 14 = (14 \cdot 15)/2 = 105$
The total number of grapefruits is 560.

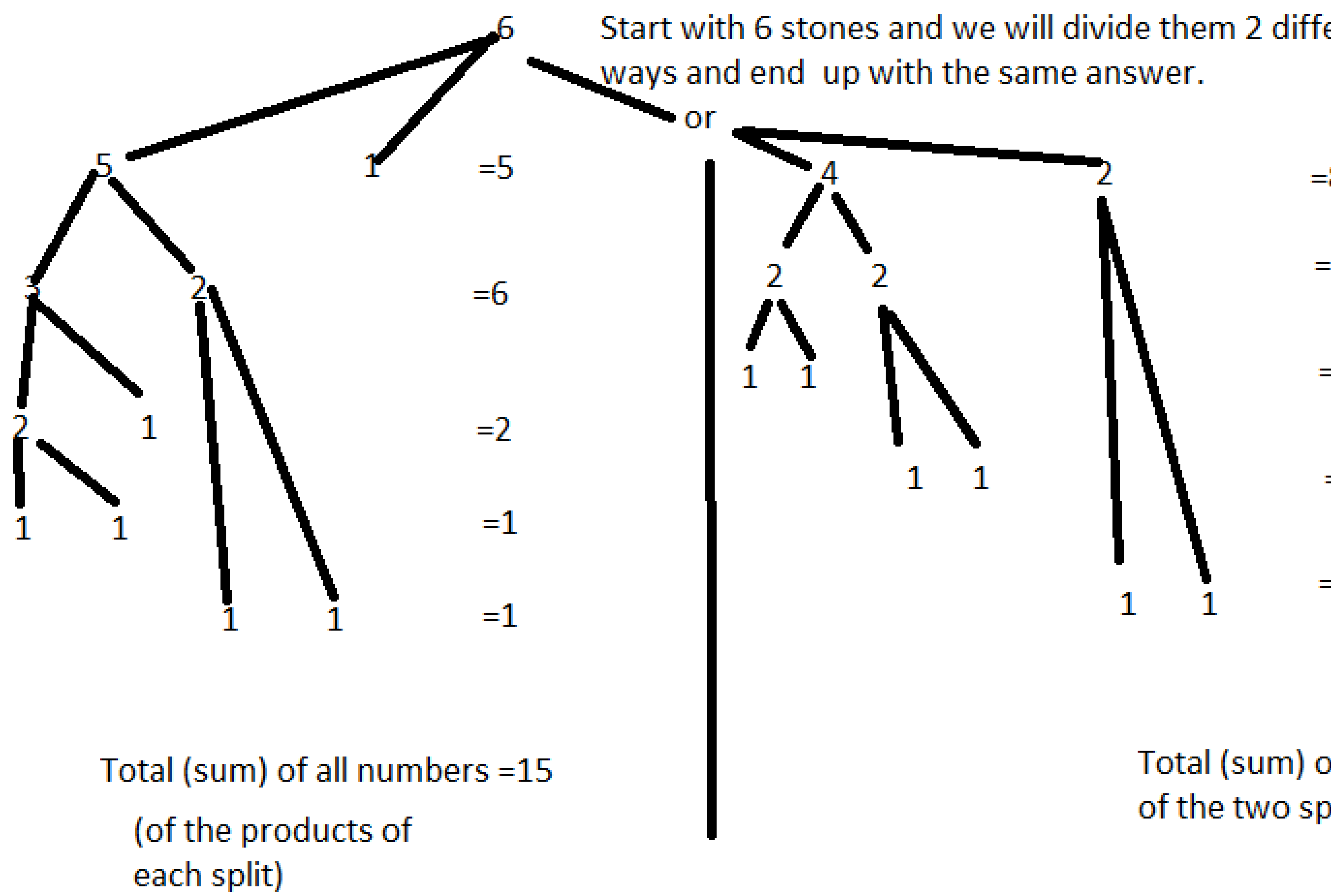
A similar problem, but in 2 dimensions is the following: If you have a group of 20 rocks, you split them into 2 groups (it doesn't matter how many are in each group) and take the product of the two piles sizes. Then, you split one of the piles into two piles (so you now have 3 piles) and take the product the two new (not all three just the 2 new groups or

piles. After 19 splits you can take the product of the two numbers (but not the two piles) . Add the products you have calculated. You would have 20 groups, each with 1 rock in it and the sum of 190. This is a triangular number 20th row with 20 numbers in it, or 20 people shaking hinds once with everyone else also would equal 190. I will show a simpler example using 10. 10->6,4 (product 24) I take the 6->4,2 (product 8), take the 2->1,1 (product 1).
Harold Reiter, " Summer 2016: Combinatorics" email handouts prior to Topics Course; Walton Plaza, Charlotte,NC, December 5, 2015



(Sum = 45 a triangular number)

Or a simpler problem, starting with 6 stones, the way they are separated doesn't matter, you will get a sum of the products equaling 15 (triangular number)



If I had split it 3 and 3 (=9); then first 3 into 2 and 1 (=2); then the 2 into 1 and 1 (=1); taking the second 3 and doing the same; the sum of the products would be 9+2+1+2+1 =15 again.

Implementing District Standards

http://www.casenex.com/casenet/frontPages/ysRC/resources/Model_Thinking_Lessons_K-6/Fifth_Grade/Math/Visualizing_the_Handshake_Problem.pdf

Bibliography

“Visualizing the Handshake Problem,” *casenex.com.*

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http://mason.gmu.edu/~jsuh4/impact/Handshake_Problem%20teaching.pdf

http://ohiorc.org/for/math/stella/problems/problem.aspx?id=323#

http://www.mathcircles.org/node/835

http://www.scholastic.com/teachers/article/court-and-its-traditions#top

http://supremecourthistory.org/htcw_justiceconference.html

Endnotes

“Visualizing the Handshake Problem,” casenex.com. http://www.casenex.com/casenet/frontPages/ysRC/resources/Model_Thinking_Lessons_K-6/Fifth_Grade/Math/Visualizing_the_Handshake_Problem.pdf.

http://math2.uncc.edu/~hbreiter

http://ohiorc.org/for/math/stella/problems/problem.aspx?id=323#

https://education.ti.com/en/us/activity/detail?id=CE05857DAFBE4447901DFD7373D58B25