Math is Golden –Exploring the Golden Ratio in Middle School

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Overview

Can middle school learners be taught the meaning of the golden ratio? I myself originally thought that this concept was beyond middle school students. I thought when I went to school I did not learn Algebra till high school. Many of the students today learn it in 7^{th} or 8^{th} grade. My brainstorm was to take a higher level concept and try to break it down and introduce it to younger students. I often use different base number systems to help with teaching exponents. Why not use the golden ratio to teach ratios and algebra?

There are a lot of middle school teachers themselves that may not be familiar with the golden ratio. This unit will break down the golden ratio and allow it to be used as a basis of learning different parts of the middle school curriculum. The importance of the golden ratio in art and history will be explored as well as its presence in nature. Students will be shown how much a part of life math is, by exploring the golden ratio.

Objectives

In this unit students will explore ratios. While exploring ratios they will become familiar with the concept of proportions, angles, side length and area. Students will construct golden rectangles. The concept of irrational numbers will be explored as it relates not only to the golden ratio but also π.

Students will also learn how to construct figures by using a compass and ruler. Students will also use algebra skills by solving problems using the quadratic formula and the Pythagorean Theorem. It is my intent that students will either learn or have reinforced many strategies and skills they need to be more advanced learners in mathematics.

Rationale

Background

I teach at an International Baccalaureate magnet school in a large metropolitan city. The student population is approximately one thousand. The student body is very culturally diverse in its make-up. Because the school is an IB magnet students are required to be "on grade level" as measured by the North Carolina end of grade assessments. Approximately 59% of the student body is identified as "TD" which stands for talent development. Math classes in middle school are at three levels, standard, standard plus and honors. For the school year 2010-2011 there are six honors level math classes in the seventh grade. Our school received the Honor School of Excellence award from the State of North Carolina for its performance last year. The one area that was not considered at the highest level was our lack of high growth. The school met growth but fell short of the high growth standard as measured by the state.

The school improvement plan this year calls for rigor, engagement and differentiation (RED). The goal is to attain the high growth that we did not achieve last year and to accomplish this with RED. Because the student body has changed and the mandate for high growth has been established I have sought out more intense higher level learning activities that will provide RED. I have to teach the curriculum but I have found I can teach it at a higher level with more engagement and with differentiation. Because my class is an honors class I am expected to get my students prepared for Algebra I next year.

My classes are all honors level math classes. The textbook I use is an eight grade level textbook and the curriculum I teach is the last level of mathematics in the Charlotte Mecklenburg School District before Algebra I. My classes consist of a fairly even distribution of boys and girls and a wide variety of students from different cultures. The cultures include European-American, African-American, Indian, African, Chinese, Korean, Vietnamese and European. In two of my classes I have students that are sixth graders and are part of the Horizons program at our school. These are students who are very academically gifted and require more advanced classes at a younger age in order to keep them engaged.

Strategies

Art

Showing students the relationship of math to art is an important aspect of the unit. Beginning the unit by showing students famous pictures that reflect mathematics will engage visual learners and show students some relationship between math and art. A website which contains some pictures of this relationship (will be used to introduce to students art in math. Some famous pictures that also use math are The Annunciation, Madonna with Child and Saints, The Mona Lisa and St. Jerome. Students will be shown these pictures.[1]

Nature

In nature there is evidence of the golden ratio as well. The spiral patterns formed by sunflowers are said to be in a ratio approaching the golden ratio. The ratio is obtained by investigating the patterns that go clockwise and counterclockwise. Most commonly the ratio is 34/55 but some with ratios of 89/55, 144/89 and 233/144 have been reported.[2] The golden ratio can also be observed in pineapples, cauliflowers and pinecones.

Pineapples usually have 5 and 8 spirals, or 8 and 13 spirals. Spruce cones tend to have 6 and 13 spirals.[3]

Ratio/Proportions

The concept of ratios will be discussed with students. With diverse levels of knowledge present in the classroom defining what a ratio is first will be important. Ratios are quotients and they show relationship between two things. Discussing with students what this means to them may allow for a deeper level of understanding of ratios.

The various forms of how ratios are written should be shown to students so that they are not confused by any later material introduced. All three methods presented below show the relationship of A to B.

First Method: $\frac{A}{B}$

Second Method: A:B

Third Method: A to B

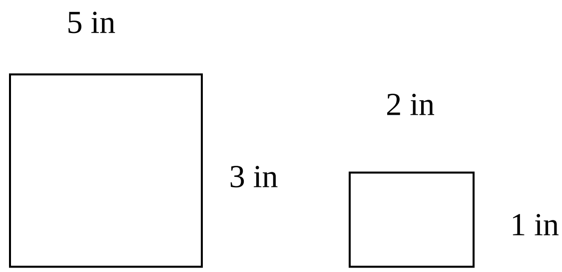
Providing examples of ratios will help students develop a higher level of understanding. Beginning with the class size and showing the ratio of boys and girls to the total number of students in the class as well as the ratio of boys to girls. Students should begin to see what conclusions can be drawn from some ratios.

Example:

If the ratio of boys to girls in the class is 3:5 and the class size is 32 then how many boys and how many girls are there?

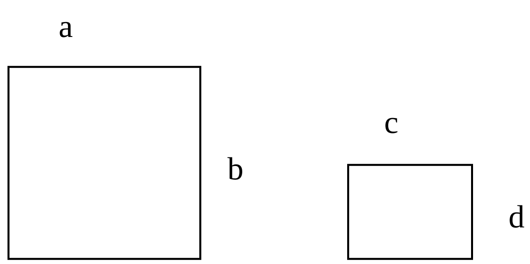
If students understand that in the ratio of boys to girls is 3:5 then there eight children present and therefore a similar ratio can be made for a size four times greater. 12:20 is in fact the same ratio as 3:5.

Using figures should be the next step in showing ratios and also the concept of proportion which shows when comparing two things there relation is the same. Referring back to the student example of $\frac{3}{5} = \frac{12}{20}$ is an example of two ratios that are in proportion. Mathematics is very strict in defining proportions. The lengths of the sides of a figure have to have the same ratio and the angles of the figures have to be congruent. It is not enough for two figures to look the same they must meet the aforementioned standards.



The two rectangles above appear to be in proportion if you were to view it without the side measurements present. Students should compare the side measurements and investigate the ratios of the sides to see if they are in proportion.

$\frac{1}{2} \neq \frac{3}{5}$ Students can investigate all the various ways of setting up the ratios and determine for themselves that the two rectangles are not in proportion.



If the above figures are in proportion then the following is true:

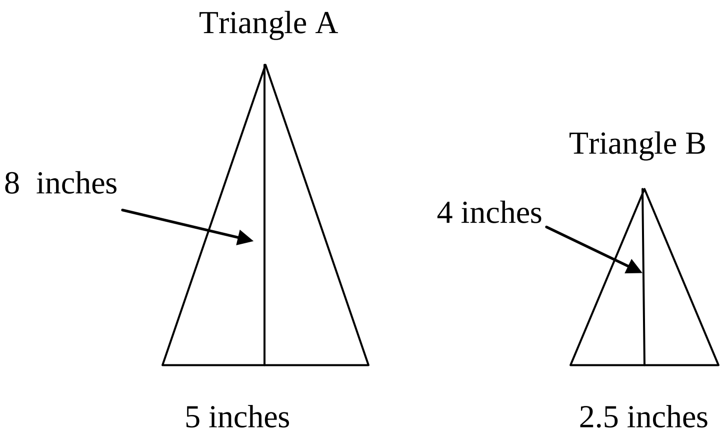
$\frac{a}{b} = \frac{c}{d}$ and $\frac{a}{c} = \frac{b}{d}$ and $\frac{b}{a} = \frac{d}{c}$ and $\frac{c}{a} = \frac{d}{b}$

It is important that students see the pattern here. When cross multiplying the same bc = ad results. It should be discussed that all comparisons either have the ratio of one side of a figure to another side of the same figure compared to the corresponding sides of the other figure. $\frac{a}{b} = \frac{c}{d}$ The other way to compare is to compare the same side of one figure to the corresponding side of the other figure and do this with both figures. $\frac{a}{c} = \frac{b}{d}$

Area/Perimeters of Figures

Students who are being taught this unit should be familiar with how to determine the area of common figures or composite figures consisting of common figures such as rectangles, squares, triangles, trapezoids as well as circles. Refreshing their memories on determining the area and perimeter of figures might be practical.

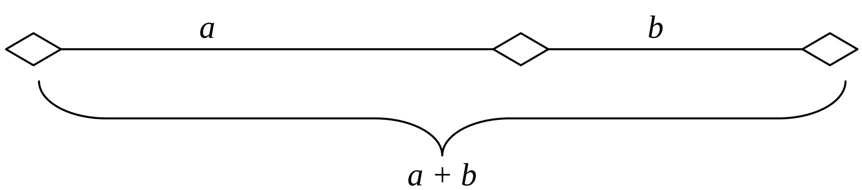
An extension of just determining the area and perimeter of figures is to determine the area and perimeter of similar figures (those whose sides are in proportion and possess the same angles). Provide students with two similar triangles. Students can easily compare the areas and perimeters of the triangles and then draw conclusions.



The area of triangle A is 20 square inches and triangle B has an area of 5 square inches yet the ratio of triangle A to triangle B is 2:1. The ratio of their areas is 4:1. When comparing perimeters students will see that the ratio of triangle A to B is 2:1.

Golden Ratio

The golden ratio is best explained by making an illustration of a line segment and showing the ratios that result which are called the golden ratio.



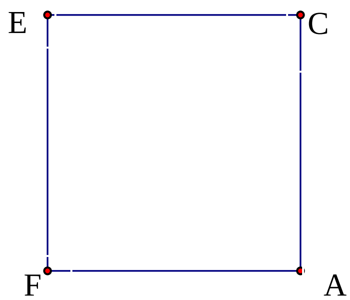
The golden ratios is when a + b is to a as a is to b or $\frac{a+b}{a} = \frac{a}{b}$. This is referred to as the Greek letter φ (phi).

Constructing a golden rectangle will allow us to give a value to phi.

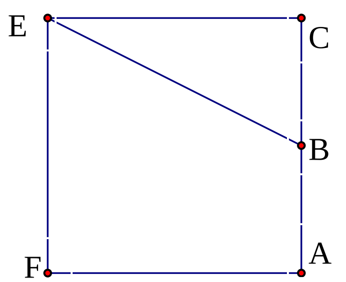
Directions:

A golden rectangle can be constructed with only straightedge and compass by this technique:

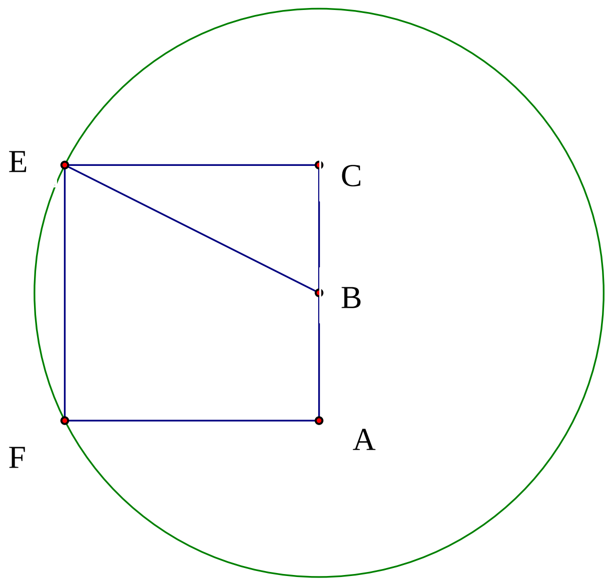
1. Students should first draw a square ACEF. This can be done by using a ruler and making sure each side is the same length.



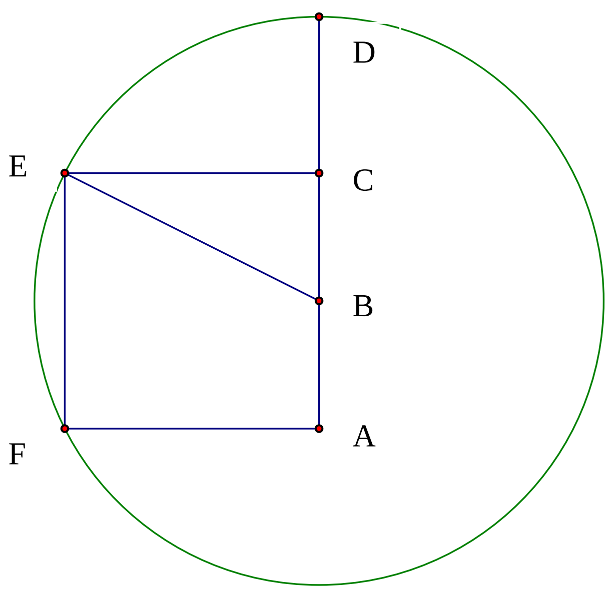
2. From the midpoint of side AC of the square they should draw a line segment to vertex E The midpoint should be determined with a ruler. Label the midpoint point B.



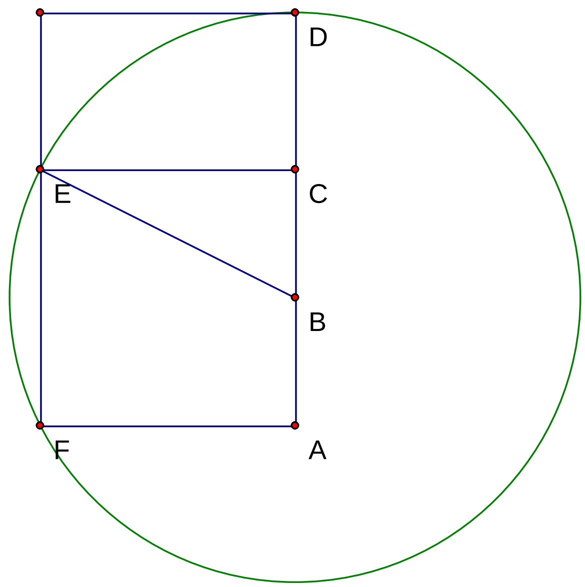
3. Using point B as the center of the circle and point E as the edge of the circle take a compass and construct a circle.



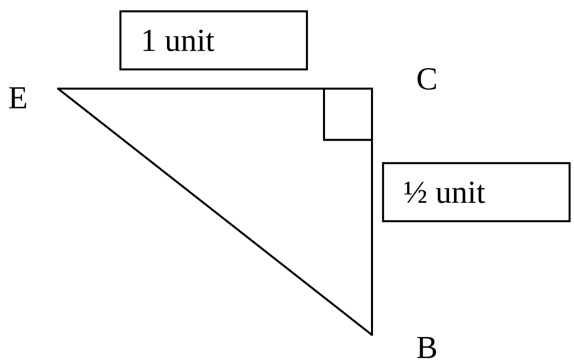
4. The rectangle can be started by using a ruler and extending side AC of the square to the edge of the circle. Label this point D.



5. The rectangle can then be finished by completing the opposite side. Use a protractor to check your angles.



In the above diagram the ratio of segment AD to AF is equal to the golden ratio! Phi can then be determined mathematically.



The goal of this construction is to obtain a value for phi (golden ratio). If $\frac{AD}{AF} =$ phi as we indicated then let's assign some numerical values to these lengths and see if we can solve determine a numerical value for the ratio or phi! Let's give the sides of the square ACEF a value of 1 unit. This makes the length of segment BC ½ unit and the side EC is of course 1. Using Pythagorean theorem $(1/2)^2 + 1^2 = 1.25^2$ or $(5/4)^2$) and the square root of 5/4 is $\frac{\sqrt{5}}{2}$. The length of segment BE to be $\frac{\sqrt{5}}{2}$. Segment BE is the radius of the circle and this means that segment BD is also $\frac{\sqrt{5}}{2}$. If we add AB to BD then we get a length of ½ + $\frac{\sqrt{5}}{2}$ and the length of AF is of course 1 so the ratio is $\frac{1+\sqrt{5}}{2} \Big/ 1$ which gives us

phi or ($\frac{1+\sqrt{5}}{2}$).

Patterns

The relationship of phi to Fibonacci numbers is very interesting[4] Students investigating the ratio of Fibonacci numbers to one another will discover the higher the numbers the closer the ratio comes to phi. This will be an investigation conducted by the students and the concept behind this investigation will be discovery.

Irrational Numbers

When phi is determined students should realize that phi is irrational. This would be a good time to explore the concept of irrational numbers. Pi is irrational and pi is derived from the ratio of the diameter to the circumference of the circle. This should be demonstrated to students and investigated with calculators. Exploring the discovery of Phi with a calculator should also help solidify the concept of irrational numbers.

Project

Students will complete the unit by designing their own piece of artwork using phi, the Pythagorean Theorem as well as pi. They must construct their artwork using all three mathematical concepts and explain the relationship of the artwork to the concepts and how they incorporated all the concepts together.

Classroom Activities

Title: Math in Art and Nature

Materials: An LCD projector, Smart or Promethean board to project internet images to students will be required.

Objectives:

- Students should be able to identify the relationship between some art and mathematics.
- Students should understand the concept of ratios.

Guided Instruction:

The teacher will visit my suggested website(s) http://britton.disted.camosun.bc.ca/goldslide/jbgoldslide.htm or they may find sites of their own that reflect pictures or art showing the golden ratio. Videos from You Tube may also be acceptable as there are many of these available. I recommend the video Nature by Numbers which can be accessed at http://www.youtube.com/watch?v=kkGeOWYOFoA. In addition to this there is a video showing the golden ratio in the human body which can be viewed at http://www.youtube.com/watch?v=085KSyQVb-U&feature=related on YouTube as well.

The pictures and videos should stimulate discussion from students. What is the golden ratio? What are Fibonacci numbers? These should be questions raised by the students. The teacher should not go into details at this point about either but the discussion, mysteries and questions should open the students up to want to learn more. The teacher should relate to the students that these questions will be addressed at the end of the unit.

The concept of ratios should first be discussed after the pictures and videos are viewed. Most students at the seventh grade level are familiar with fractions and some might know that a fraction is a ratio. The teacher should explain that ratios can be represented in various ways. The first way a ratio can be written is as a fraction, the second way is in words such as three to five and the third way is written with a colon separating them as in 3:5.

Ratios that are equivalent are said to be proportional. The concept of proportions should be explained to students. Teachers should use figures such as triangles, rectangles, and even circles to illustrate proportions.

Title: Area of shapes and the Pythagorean Theorem

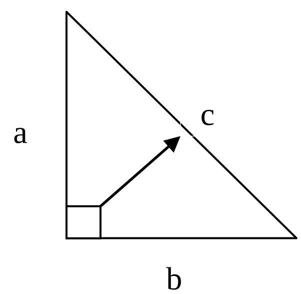
Objectives:

- Students should be familiar with how to determine the area of various shapes.
- Students should know how to use Pythagorean Theorem to find the missing lengths of a right triangle.

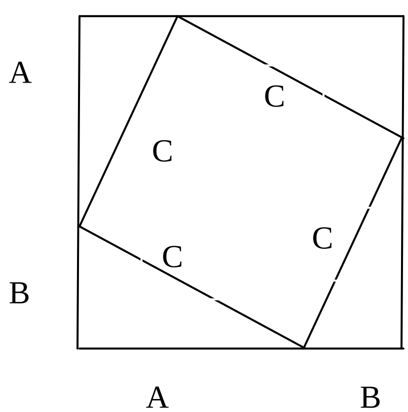
Guided Instruction:

Students will review the methods and formulas for finding the areas of various shapes. They will then discuss Pythagorean Theorem.

The Pythagorean Theorem is $a^2 + b^2 = c^2$ which simply states that the length of one leg of a right triangle squared added to the length of another leg squared is equal to the length of the hypotenuse squared. The hypotenuse is the side of the right triangle directly across from the right angle (90^0).



Teachers can use any proof they would like to help understand the Pythagorean Theorem better but I would like to use the Four Triangles in a square with length (a + b) proof.



$$(A+B)^2 = 4(\frac{AB}{2}) + C^2$$

$$A^2 + B^2 + 2AB = 2AB + C^2$$

$$A^2 + B^2 = C^2$$

Completing this proof will help prepare students for some of the math needed to solve for the golden ratio in later instruction.

Teachers can give students various problems involving the Pythagorean Theorem to practice solving for leg lengths and the hypotenuse.

Title: Golden Ratio

Objectives:

- Students will mathematically arrive at the golden ratio through construction of a golden rectangle.

Guided Instruction:

The teacher should first draw a line segment and then segment it about ¾ of the way along. The teacher should then explain the golden ratio using just variables. Once students are comfortable with this then the teacher should pass out materials and have students construct a golden rectangle using a ruler, compass and a pencil. Following the instructions listed in strategies students will arrive at the mathematical representation of the golden ratio.

Students can also construct a golden triangle. An isosceles acute triangle is a golden triangle if the ratio of the length of the side over the length of the base is the golden ratio.

Title: Fibonacci and the Golden Ratio

Objectives:

- Students will learn what Fibonacci numbers are.
- Students will discover the relationship of Fibonacci numbers to the golden ratio.

Guided Instruction:

The teacher will explore Fibonacci numbers with students. By taking Fibonacci numbers and placing them in ratios the students should discover that the larger the Fibonacci numbers become the closer to the Golden ratio their ratios become.

Fibonacci numbers were named because the terms appear in Fibonacci's *Liber Abaci* in a problem about rabbits.[5] The Fibonacci numbers start with the numbers 0 and 1. These numbers are added to obtain the next number of 1. The last two numbers in the sequence are added to give us the next number of 2. The Fibonacci numbers are 0,1,1,2,3,5,8,13,21,34,55,89,144,233,377 and so on.

The Fibonacci numbers relate to the golden ratio by taking the ratio of consecutive numbers and comparing them to phi.

Example: $\frac{1}{1},\frac{2}{1},\frac{3}{2},\frac{5}{3},\frac{8}{5},\frac{13}{8},\frac{21}{13},\frac{34}{21},\frac{55}{34}\frac{1}{1},\frac{2}{1},\frac{3}{2},\frac{5}{3},\frac{8}{5},\frac{13}{8},\frac{21}{13},\frac{34}{21},\frac{55}{34}\ldots$

Student should feel free to use a calculator for determining the values of the ratios. If we take determine the ratios up to the nearest thousandths we get:

1, 2, 1.5, 1.667, 1.6, 1.625, 1.615, 1.619, 1.618…..

The values of these terms get closer to phi, the golden ratio.[6]

Title: Exploring irrational numbers

Objectives:

- Students should become familiar with irrational numbers and the concept of what an irrational number is.

Guided Instruction:

The teacher will discuss pi and phi as irrational numbers that students are familiar with. The teacher will demonstrate a simple proof to show why certain numbers are irrational.

Proof:

Suppose $\sqrt{2}$ is rational then $\sqrt{2}$ = p/q and p/q is expressed in it lowest terms. If we square both sides then the result is $2= p^2 /q^2$. If we solve for p then the result is $2q^2=p^2$. We know that p^2 has to be even because q^2 is multiplied by 2 and any number multiplied by 2 will be even. So we know we can divide by 2. It would look like this:

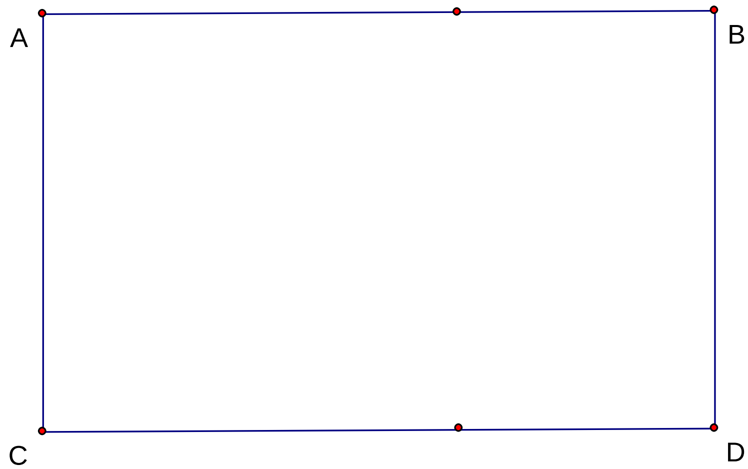
$q^2 = p/2\ (p)$

We can then conclude the q^2 is also even meaning that q is even. The conclusion we have to draw is this is not possible because p and q are in their lowest terms and therefore 2/2 does not equal the $\sqrt{2}$.

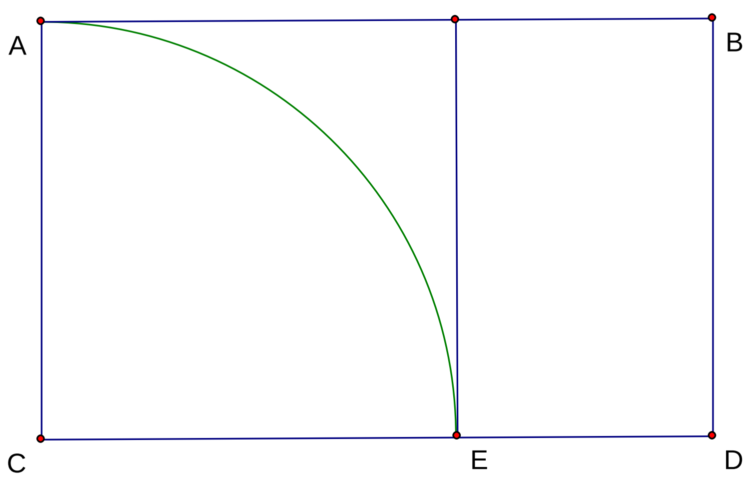
Project: Artwork involving phi or pi.

The teacher will pass out a rubric to students explaining the work that will be required in order to successfully complete the project. The main idea is that students will create a piece of artwork using the concepts discussed in the unit. The golden spiral is a construction that students could do and then color in based on their own preferences.[7]

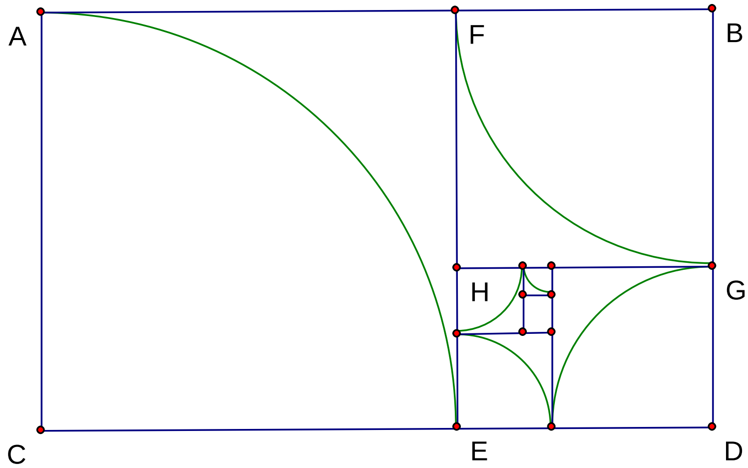
Start with a golden rectangle:



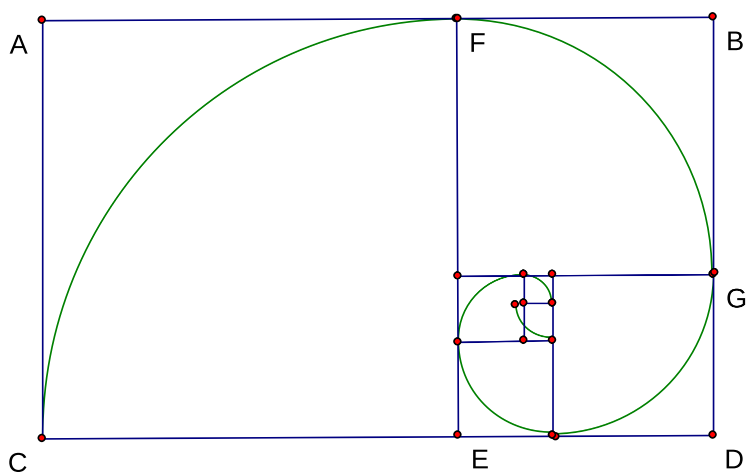
Construct square as shown



You continue this process by making circles inside the smaller and smaller golden rectangles making smaller and smaller squares.



Then inscribe quarter circles in each as shown. The first quarter circle is centered at E and has a radius CE.



Students could make drawings of sunflowers, pinecones, pineapples, cauliflowers or other examples in nature that reflect the Fibonacci numbers and thus the golden ratio. They could also investigate famous paintings that have the golden ratio and draw something reflecting what they learned such as a self portrait or building using the golden ratio.

Notes

[1] Jill Britton "Homepage" http://britton.disted.camosun.bc.ca/goldslide/jbgoldslide.htm)
[2] Mario Livio, The Golden Ratio (New York: Broadway Books, 2002), 112.
[3] Alex Bellos, Here's Looking at Euclid (New York: Simon and Schuster, 2010), 197.
[4] Mario Livio, The Golden Ratio (New York: Broadway Books, 2002), 101.
[5] Alex Bellos, Here's Looking at Euclid (New York: Simon and Schuster, 2010), 197.
[6] Alex Bellos, Here's Looking at Euclid (New York: Simon and Schuster, 2010), 200.
[7] Sasho Kalajdzievski, Math and Art An Introduction to Visual Mathematics (Boca Raton: CRC Press, 2008), 18.

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Bellos, Alex. *Here's looking at Euclid: a surprising excursion through the astonishing world of math*. New York: Free Press, 2010. Print. I used this book to cite specific inforamtion regarding the appearance of the golden ratio in nature.

"Jill Britton Homepage." *Jill Britton Homepage*. Jill Britton, 8 July 2010. Web. 30 Sept. 2010. <britton.disted.camosun.bc.ca/goldslide/jbgoldslide.htm) >. Utilized this website to give some visual background to the realtionship of the golden ratio to art.

Kalajdzievski, Sasho, and R. Padmanabhan. *Math and art: an introduction to visual mathematics*. Boca Raton: CRC Press, 2008. Print. I used this book to obtain the constructions of golden triangles and golden spirals.

Livio, Mario. *The golden ratio: the story of phi, the world's most astonishing number*. New York: Broadway Books, 2002. Print. I used this book to obtain a basic understanding of the golden ratio as well as its history. I cited the book for specific references to the realtionship of the golden ratio to the Fibonacci numbers as well as the appearance of the golden ratio in nature.

SpaceRip. "YouTube- Nature by Numbers ." *YouTube- Broadcast Yourself.* . N.p., n.d. Web. 26 Nov. 2010. <http://www.youtube.com/watch?v=kkGeOWYOFoA>. Used this site to show the realtionshio of the golden ratio to nature.

proussakis. "YouTube- Golden Ratio in Human Body ." *YouTube- Broadcast Yourself.* . N.p., n.d. Web. 26 Nov. 2010. <http://www.youtube.com/watch?v=085KSyQVb-

U&feature=related>. Used this video to show the relationship of the golden ratio in art, nature and the human body.

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