

Math and Pop Culture – Count on It!

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Introduction

I teach seventh grade mathematics in an urban school located in Charlotte, North Carolina. My school was classified as a full International Baccalaureate magnet school two years ago and it resulted in a change in the spectrum of students enrolled at our school. All students must enter a lottery system and then be selected to attend our school. They must also be on grade level as it relates to their end of grade scores in mathematics and reading. The school has a diverse student body with students from all over the world represented in my classroom. I have been teaching seventh grade mathematics for eight years and I still feel like I am a new teacher. I constantly seek out better strategies and a deeper level of understanding of my content area. I typically have few to any discipline problems in my classroom and the students work well in groups. I always keep in the back of my mind however that students are effective for only short periods of time in small groups. I have to refocus and direct them on a periodic basis.

The Unit that I have created will help students discover the joys of counting. Not the simple 1, 2, 3... kind of counting but knowing how to count multiple events and determine the mathematical probability of something occurring by knowing how to count. My intent in writing this unit is to take students from a very basic level of understanding how to count things to a higher level. I want to teach the methods that lead to the development of formulas. If you are going to teach this unit then this must be an important part of your instruction. Providing formulas and allowing students to “plug” in numbers and solve does not give the deeper level of understanding that I am attempting to reach.

The unit is intended for a middle school classroom and may be differentiated for an elementary class or high school class with some success. The area of mathematics that this unit primarily deals with is called combinatorics. The new Common Core Standards have moved the areas of probability, combinations and permutations to seventh grade. With this in mind I would have to say that since I teach seventh grade and the Common Core Standards address these areas in seventh grade that this would be the best grade level for this unit.

Rationale

I have chosen to write this unit on combinatorics because this is my weakest area of mathematics. I am strengthening my skills in this area through the development of this unit. By researching and explaining how to approach the variety of problems I have presented I hope to become more adept at solving the problems myself. I know that an area of concern that I possess is recognizing when a problem is a permutation or combination? Is there over counting present or not? Being taught formulas without conceptual understanding hurts more than it helps. I avoid teaching formulas to students in all my instruction as I feel it is vitally important that students discover the formula for themselves and why it works before they use it. In combinatorics I think this is especially important.

My students currently learn about permutations, combinations and counting when they are in the sixth grade as indicated by the 2003 North Carolina Standard Course of Studies (NCSCOS). North Carolina is moving to the Common Core Standards next year and these areas of mathematics will move from the sixth grade to the seventh grade. I want my students to be competent in these areas and be able to pass Common Core assessments that are not multiple choice but require problem solving skills at a high level.

Factorials

Students who are not familiar with factorials should be introduced to the concept prior to starting the unit. Factorials are written with the number and then an exclamation mark. The value of $3!$ is determined by multiplying $3 \cdot 2 \cdot 1$. The value of $3!$ is 6. When dividing factorials we can simplify numbers to make it a much easier process. The problem $6!$ divided by $4!$ is actually 720 divided by 24 which of course is 30. To make it easier we can write these out as $\frac{6!}{4!}$ which can be simplified by recognizing that: ¹

$$\frac{6!}{4!} = \frac{6 \cdot 5 \cdot \cancel{4 \cdot 3 \cdot 2 \cdot 1}}{\cancel{4 \cdot 3 \cdot 2 \cdot 1}}$$

Thus $6!$ divided by $4!$ is simply 6 times 5 since everything 4 and below is reduced to one. It might be easier to show students that $6!$ can be rewritten as $6 \cdot 5 \cdot 4!$. This would mean that if it was being divided by $4!$ then $4!$ is being divided by itself and thus is one.

Counting Lists

Have you ever wondered how many numbers are there from 3,011 to 10,678? What if we needed to know? I think many people would say that there were 7,667 numbers because they would subtract. Where is the mistake in that? We miss a number. Take a look at 1 to 3 and count on your hand. What? There are three? Of course 1, 2, 3, not two as we might

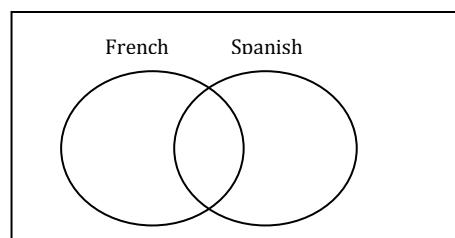
have gotten had we followed the same logic as before. Lists of numbers can come in all different forms. What if I needed to know all the multiples of 7 in the interval 93 and 798? How could I do that quickly and efficiently? First I would find the lowest multiple of 7 which is 98. Next I find the highest multiple of 7 in the interval which is 798. A key is noticing that 7 goes into 98 a total of 14 times and into 78 a total of 114 times. This results in a list 14 to 114. We would like our list to be as easy as that 1, 2, 3 problem so how can we do it? If we subtract 13 from every number, the first number becomes a 1 and the last 101. So there are 101 multiples of 7!

This same philosophy holds true for sequences like 33, 35, 37...569. How many numbers appear in this list? Note that this is the same as asking how many odd numbers exist between 23 and 570. To answer this equation, we first will add one to each number which produces a list of even numbers. Adapting our earlier technique, we divide each number in our list by 2 and produce the list 17, 18, 19...285. After subtracting 16 from each number, we find that our original list had of 269 odd numbers.²

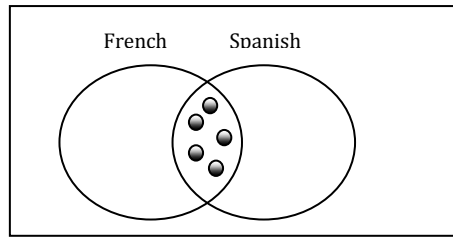
The overarching idea that the teacher needs to get across to students is that we always find a pattern and then make it simple.

Counting by Addition or Subtraction

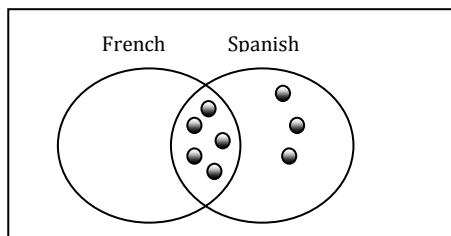
Counting by addition or subtraction can be accomplished through the use of Venn diagrams. Venn diagrams were named after John Venn an ordained priest from England who published his first diagram in 1881.³ We have a group of twelve middle school students who are on the Math Club team and all of them are taking at least one foreign language class. The school offers Spanish and French as its foreign language classes. Eight of the players are taking Spanish and five are taking both languages how many players are taking French? We would create a Venn diagram:



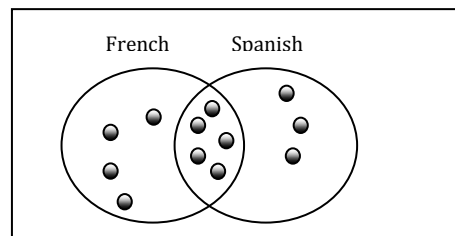
We could place points in the circle to represent the players. A point that is in the French circle that is not in the Spanish circle is for students who are taking French only. The same holds true for points in the Spanish circle only. Points that are in both circles are for students taking both foreign languages classes. A point that is not in either circle is for students that are not taking any foreign language classes.



This figure represents the five players taking both foreign languages. Since I know that there are eight players taking Spanish and five are taking both then that leaves three left taking Spanish only.



I place the dots representing those three players into the diagram. I know that since we have twelve total players and only eight are shown here and there are no points outside the circle because all the players are taking a foreign language class then there we add the three and the five to get eight and that must mean there are four players taking French only.



The above diagram reflects the distribution of players that matches our problem.

Fundamental Counting Rule

This rule describes the basics that need to be understood by students. The idea is that the first event or what you are counting occurs in a certain number of ways and then the second thing you are counting occurs in a certain number of ways and on and on until the last one which is the number of things you are counting. This is given as $k_1 \cdot k_2 \cdot k_3 \cdots k_n$ which if first shown to students means absolutely nothing.⁴ I would explain it by using an example such as ice cream. If there are three choices for how to get it like a cup, one

scoop cone or two scoop cone and let's say we have eight flavors then how many ways can we get our ice cream? The answer is simple. We can get it served three different ways and eight flavors so 24 possible combinations. We would have to assume that the two scoops did not have two different flavors for this to work.

Permutation Rule

This is the rule that has to be used when the order of objects matters. This can be confusing to students. I would like it to be explained in its simplest form that when I am choosing something then it cannot be chosen again. For example if I had nine distinct books and I wanted to know how many ways they could be set up then I would start with the fact that I have nine books then once I choose one I have a choice of eight books left for the second book all the way to the last book where I will only have one choice. This can be easily shown in factorial by writing $9!$

If we were to use the above example with the books however and we wanted to choose only three books to set up out of the nine we know have a different problem. This could be simply explained to students that instead of doing $9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ we only need to do $9 \cdot 8 \cdot 7$ since we only choosing out three books and not all nine. There is a formula for this as you can imagine. The total number of things we have is represented by the variable n . How many we are choosing from among that total number is represented by r . We have the $n!$ divided by the total number of things we have minus what we are choosing $(n-r)!$ ⁵

Combination Rule

A combination is when a collection of things are chosen without regard to the order. A good way of explaining this would be to show the difference between a permutation and a combination. Ice cream is always good to use. So, what if we have four flavors of ice cream? They are vanilla, chocolate, strawberry and peach. If we were choosing flavors for a two scoop cone in a permutation vanilla and chocolate would be one possibility and then chocolate and vanilla would be another possibility. This is because order matters and thus makes the selections different. If we did not care about the order then vanilla and chocolate could not be counted with chocolate and vanilla in a combination situation. A list might demonstrate this more effectively.

Permutation outcomes: VC VS VP CV CS CP SV SC SP PV PC and PS

Combination outcomes: VC VS VP CS CP SP

The formula for combinations is actually more difficult because we have to account for all the over counting or duplication that occurs because of the repeated variations of ways to get things and eliminate all those.

In the above example we had 4 items or $n=4$. We were choosing 2 things from this group but order did not matter. We use the same formula as a permutation with one addition ———⁶ we had to divide by the factorial of the number we are choosing times the factorial of the number we did not choose.

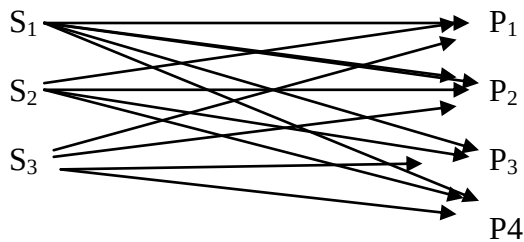
Counting using the Multiplication Rule

We know that multiplication is simply repeated addition. We teach students this at an early age. We can use this concept to help us count things that otherwise would take an incredible amount of time and effort to do. We take the eight students to an ice cream shop that randomly dispenses either vanilla or chocolate ice cream when you pull the lever on the machine. How many different outcomes of ice cream could appear atop the cone? Here is one VCCCVCVV where the V represents vanilla and the C chocolate. This is fairly easy to explain as there are only two possible outcomes each time a student dispenses the ice cream, it is either vanilla or chocolate. Since we are doing this eight times and there are two possibilities each time then the problem can be solved by multiplying 2 times 2 eight times or 2^8 . This rule states that *when something takes place in several stages, to find the total number of ways it can occur, multiply the number of ways each individual stage can occur.*⁷ A formula that could be used would be to count the number of sequences of k objects chosen from a collection of n objects and represent it as n^k .

The next situation we have to concern ourselves with is when we choose sequences that cannot contain the same things. If we were choosing three numbers from among twenty five numbers but we could not repeat them then we would have twenty five options to begin. Then, we would have twenty four choices (since one is no longer an option). Then, we finally have twenty three remaining possibilities. So, the total number of possible outcomes is $[(25)(24)(23)=13,800]$. Representing this as a formula is a little more difficult. We start with n times $n-1$ and so on for k times. So if we look back at our example the 25 is the n and we are choosing three numbers so 3 is k . Our formula has to be $n(n-1)(n-2)\cdots(n-k+1)$. The first number being multiplied is the amount of numbers chosen subtracted from the first number plus one since we have counted the first number already.

Counting Multiple Events

Counting multiple events helps us to use a strategy that I will focus on in this unit. Discover the pattern present and use simple examples to solve larger problems. When trying to count multiple events we can use casework or not use casework. When presented with a small problem like “if we have three shirts and four pairs of pants how many outfits consisting of one pair of pants and one shirt can we make?” We can just list them in this problem we have three shirts and four pants then we could have the following possibilities:



If we count all the possibilities we clearly have three shirts times four pants which gives us twelve possible combinations. This works fine when we have so few to work with but what if we have a large number or more than two things to choose from?

If we were to take a wardrobe problem like the one above and make it more difficult we could still use the same strategy. What if we were packing for a trip and we packed four pairs of shoes, two pairs of shorts and four pairs of pants as well as four shirts and three t-shirts along with two hats? Assuming there was no color problems and everything matches how many different wardrobes could we have if we had to cover our upper body, our lower body, our feet and our head? To solve this drawing a diagram might help.

If we get dressed in stages, like most of us do it might go like this:

First Stage (Lower body)	Second stage (Upper body)	Third stage (Feet)	Fourth stage (head)
2 shorts + 4 pants = 6 ways + no hat = 3	4 shirts plus 3 t-shirts = 7 ways	4 pairs of shoes	2 hats

The math would be $6(7)(4)(3) = 504$ ways to be dressed.

What if we had a problem where we wanted to form a committee of four with one teacher from each subject form among three algebra classes, two geometry classes, one general math class two U.S. history classes, two World history classes, two biology classes and one Chemistry class and seven language arts classes?

There are six math classes to choose from and four social studies so we have twenty four possible combinations from these two classes then there are three science classes so we multiply twenty four by three for seventy two ways then again by seven to account for the language arts classes. This leaves us with five hundred and four possible ways to make a committee.

When choosing the clothes combinations and the committees we were choosing from among things that were not related to each other or did not depend on the other. These decisions are called independent decisions since choosing one thing does not affect the other. In the first problem I could choose a shirt and then any pair of pants I wanted. The shirt I choose did not affect the pants I then choose. The same is true when I was forming the committees. The teacher I choose from one subject did not affect the teacher I choose from the other subjects. Our strategy for solving independent problems then is to simply multiply all the possibilities together.

Events that are not independent are handled in a different manner. A problem as simple as determining how many ways four books can be arranged on a shelf is clearly not independent as once I pick a book then this changes the number and kinds of books I am left with to choose from. We cannot just do four to the fourth power and have our answer. We first will have four books to choose from and then once we make our choice we will only have three to choose from then two and finally just one book left. The math is four times three times two times one which most will recognize as $4!$ or four factorial.

These types of problems occur whenever we have to choose several items one at a time from a larger group of items. For example let's say that it is Halloween and we have scored it big with twenty different pieces of candy! Our parents know that should we eat all this candy at one time we will get sick. They allow us to choose three pieces of candy from the twenty we got but there is a condition, we have to do it from a closed bag. How many ways can the first three pieces of candy be chosen? Since this is a permutation there are twenty ways to choose the first piece, then there are nineteen ways to choose the second and eighteen ways to choose the third and that is it since we only get to choose three pieces. If we multiply then we get $20(19)(18) = 6,840$ ways to get the first three pieces of candy. Just think if we had that many pieces! That would be a real stomach ache!

Taking this concept to the next level would be if we wanted to get more than one result? What if we stayed in our ice cream store and decided to order a cone with two different types of ice cream on it and we had twenty flavors to choose from? Would the answer simply be to multiply $20(19) = 380$? This gives us three hundred possible ways to get two different flavors from twenty possible flavors. The problem here is that if we got chocolate and then vanilla it is the same as getting vanilla and then chocolate so we are actually over counting the possible results. The only to solve this is to divide by two. This is because we have only two ways of getting vanilla and chocolate. This means we have

150 possible ways to get a cone with two flavors of ice cream on it from a choice of twenty flavors.

If we went to three flavors does it make sense to divide by three? We have to look at this because remember I said that we divided by two because there were only two ways of getting chocolate and vanilla. It was either VC or CV. What if we added strawberry and we were looking at three toppings? Do we only have three ways of getting chocolate, vanilla and strawberry? If we do this out in case work we see we can have VCS, VSC, SVC, SCV, CVS, and CSV as possible outcomes. That is more than three it is six. I think I see a formula being developed here. We can write this as a fraction where we do the numerator using the multiplication rule and the denominator as the number of ways to get one outcome. In this problem we have twenty flavors and three toppings. This would be:

$20(19)(18)/3(2)(1)$ or $6,840/6=1,140$ possible ways to get your three flavored cone.

Since we are eliminating all possibilities of getting the same combinations we should be able to develop a formula for this situation. We can use factorials and say that $20!/(20-3)!$ divided by $(3!)$ or $n!/(n-r)!$ divided by $r!$ works here.

In the above example we treated all combinations of vanilla, chocolate and strawberry as the same thing. If we did not worry about this then we could develop another formula involving factorials. If we have twenty choices of ice cream but we are only choosing three then we have 20! to start with don't we? We are then limiting our choices down to three. If we make our first choice then there are nineteen choices and then we will make our last choices from among the eighteen choices left. We could simply use the formula $n!/(n-r)!$ because we do not need to worry about over counting.

Probability

Counting leads to probability. Knowing how to count and determine outcomes leads you to be able to determine the probability of an outcome. Starting off with a coin is an easy way to explain probability. There are only two outcomes, a heads or a tails. What if we wanted to flip a coin six times and we wanted to know the probability of getting three heads and three tails? We know that there are 64 outcomes because we have two possibilities each time for six times so 2^6 as we discussed earlier with the multiplication principle. The probability of getting any one outcome is $1/64$. The question is how many of these outcomes have exactly three heads and three tails? Since we have six coin tosses and if we get three heads we know we get three tails or vice versa then we can simply do $6!/3!(6-3)!$ This is also known as 6 choose 3. This gives us 20. Since there are 20 combinations of three heads and three tails. To solve this then we are left with twenty out of sixty four combinations that have three heads and three tails.

We can also use the subtraction principle with probability. What if we were to flip a coin eight times and we wanted to know the probability of getting at least three or more heads? We know from the multiplication principle that we have 2^8 outcomes or 256. In this situation it is easier to determine the complement or how many times we do not get three or more heads. To describe this we will add up these possibilities first no heads would be 8 choose 0 plus one possibility of 8 choose 1 then two possibilities so 8 choose 2. This is $1 + 8 + 28$ or 37 outcomes where there are no heads, or one or two heads present. If we subtract this from the total number of outcomes we get 219 outcomes that at least three heads are present so our probability becomes 219 out of 256 or about 85%.

Activities

Gauss

Most mathematicians and mathematics teachers know the story of Gauss. I think it is important to share this with students. Giving students a view of mathematics from an historical perspective is important. Students should know that someone somewhere came up with the material that they are currently learning. Using the story of Gauss and having them discover the formula for counting an arithmetic series will be more effective than just giving them the formula in their notes.

The story goes that Gauss was in the second grade and his teacher asked the class to add all the numbers from one to one hundred. Within minutes Gauss raised his hand and said he was done. The teacher was amazed by his speed and by the fact that he had the correct answer. When asking your students to do these please note those that start with $1+2$ and $3+4$ and the $5+6$ and so on. What Gauss saw was the total number of numbers in the list of numbers from one to one hundred. That number is obviously one hundred, he then noted that adding the first and the last gave a total of 101 and the second and the second to last numbers also totaled 101. He observed that these were pairs of numbers so there were obviously 50 pairs so $50(101)$ gave the answer of 5,050. He then developed a formula to help him do this for any list of numbers created with an arithmetic sequence. Have your students try to develop their own formulas that will work for different lists of arithmetic sequences.

In addition to completing the above work have students write a short few paragraphs on the whole story of Gauss from another student's prospective. These could be quite humorous and shared with class. This idea was obtained from Dr. Tim Chartier and his wife Tanya's workshop called Acting out!-(Using Creative Drama).

Lottery

The State of North Carolina has the Education Lottery and proceeds from this lottery are used to fund education in the state. No teacher should advocate gambling but the connection of math to the lottery is very obvious. A fictitious lottery could exist where a person could pay \$1 and then choose six balls numbered from one to fifty where order does not matter. If you guess all six then you win the jackpot. Math comes into play here by first determining how big the jackpot would have to be if you could place every possible bet and win! Next the number of bets you would have to place in order to guarantee winning.

If we take a small example of say only picking two balls from five numbers we can demonstrate much easier how this works. In our game getting a ball with a one on it and then a five is the same as a five and then a one. Order does not matter so we can call this a combination game.

Let's list all the options:

1, 2	1, 3	1, 4	1, 5
2, 1	2, 3	2, 4	2, 5
3, 1	3, 2	3, 4	3, 5
4, 1	4, 2	4, 3	4, 5
5, 1	5, 2	5, 3	5, 4

If we look at this we must remember that 1,2 is the same as 2,1 so how many times is pair listed? Obviously each is duplicated twice so we have ten options. It might be interesting to see if the students see the pattern here.

1, 2	1, 3	1, 4	1, 5
2, 1	2, 3	2, 4	2, 5
3, 1	3, 2	3, 4	3, 5
4, 1	4, 2	4, 3	4, 5
5, 1	5, 2	5, 3	5, 4

One side is mirrored on the other side.

So why not put this in a math form we may be familiar with?

$5 \times 4 / 2 \times 1$ which gives us the ten possibilities.

The concept here is to use the pattern about what we know for an easy example and extend that same idea to a more complex and difficult problem. In our case the problem of picking six numbers from a set of fifty numbers numbered 1-50. We would have to count all the possible collections of six numbers and divide by how many times each unordered collection was counted.

$\frac{50 \times 49 \times 48 \times 47 \times 46 \times 45}{6 \times 5 \times 4 \times 3 \times 2 \times 1}$

This is exactly what we did with the easy version of our lottery game. We are now working with much bigger numbers however!

Coins

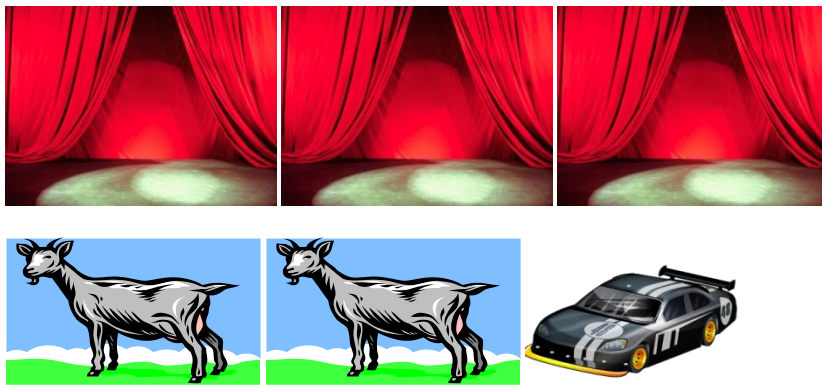
Have students take what they learned about coins and decide if the following game is something they should play and if one person has a better chance of winning than the other. They should determine the probability to decide which way gives the best chance of winning.

What if we played a game where you flip nine coins and if they come up 4/5 or four heads and five tails or five tails and four heads you lose and pay one dollar? Could you figure out the math and determine if this was a good game to play?

If we determine all the outcomes we get 2^9 or 512 possible outcomes. The probability can be determined by adding 9 choose 4 and 9 choose 5 which is 252. That means that the chances of getting these outcomes are 252 out of 512 or right at 49%. You would have a 51% chance of getting combinations that did not have the 4/5 split. It might be a game you would want to play.

Have your students come up with their own game involving coins and prove, by using probability, which strategy would be the best to play.

Monte Hall Problem



This problem is very famous among mathematicians and teachers alike. I first heard about it in a graduate level class at the University of Charlotte taught by Dr. Adam Harbaugh and Dr. David Pugalee. The problem relates to an old game show called “Let’s Make A Deal” that was hosted by Monte Hall. The idea was to get a contestant to pick a prize behind a door but to confuse them for entertainment purposes. Monte would explain that they had a chance to win something on the level of a new car or expensive trip if they choose the right door out of three offered. If they choose incorrectly then they would win something like a goat.

The contestant chose a door and Monte then proceeded to say how great their pick was because behind one of the other doors was a goat, and he would open the door revealing the goat. The crowd would cheer and the contestant was excited, which made for great television. Monte would then ask the contestant if they were happy with their door or did they want to switch their choice to the remaining door. The crowd would yell out change or switch depending on their individual logic. I would of course yell out a mathematician’s choice because I knew that one of the choices offered a better chance of winning, or did it?

Explain this problem to your class and search websites that have apps with great illustrations and are interactive. Students should learn that they need to switch. The logic of it is fairly simply. In the beginning we are given a $\frac{1}{3}$ chance of winning and a $\frac{2}{3}$ chance of losing. If we were given the chance of $\frac{2}{3}$ to win and $\frac{1}{3}$ to lose what do you think most people would do? That is what happens when Monte gives us the chance to switch to the remaining door. He has opened one of the two doors we did not choose, revealing a goat, and we get to choose the remaining door thus a $\frac{2}{3}$ chance that we would have jumped at if we were offered it at the beginning of the game. Students will argue the 50/50 chance because we have one door out of two however we need to go back to the beginning to explain the problem. The sample size is important so using websites that allow many chances will clearly shows that you win more when switching doors.

Subway



I have pasted in some pictures that are current Subway options for purchase at their restaurants. A nice classroom activity that students could participate in is to determine all the combinations that can be created at Subway given the above options. Students could be asked how many ways I could order my meal? It should be made clear that they can have a flatbread sandwich, a six inch sandwich, a foot long sandwich or a salad.

The math is counting by multiplication rule so we have four choices and eighteen ways to make each choice. The total number of combinations is a staggering 68,719,476,736 or

4¹⁸. Students should discuss if this is really accurate and explore some combinations. A great way to demonstrate this is by having students create posters showing the number of combinations that can be created in a Subway store. Students should be creative in exploring this. I would think that working in small triad groups might work the best for this.

¹ (Bluman 2005) P. 572

² (Patrick 2007) P. 5

³ (Patrick 2007) P.7

⁴ (Bluman 2005) P. 570

⁵ (Bluman 2005) P. 574

⁶ (Bluman 2005) P. 578

⁷ (Tannenbaum 2004) P. 595

Works Cited

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I used this book to obtain the different rules such as counting, permutation and combination.

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