Backwards Numbers: A Study of Place Value

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Introduction

I was reaching for my daily math sheet that my school has elected to use and in big bold letters in a box it said: TO ADD NUMBERS YOU MUST START ON THE RIGHT SIDE… I paused and asked the class if that was true. They looked at me as if I'd asked if the sky was blue. "Of course," they replied, "it's true". "You must start on the left side with the numbers in the ones column." When I asked them why they thought this they could provide me no explanation that would prevent them from starting in a different place value column.

When I proposed that I could start anywhere I wanted and showed them a three digit by three digit addition problem, they were amazed - and they wanted to try it! They loved it, but some said it was harder than the "other way". I asked, "How many times have you done it starting on the right or in the middle?" "Twice," they replied. How many times have you done it your way?" "Thousands," came the answer.

Overview

I teach fourth grade at an elementary school comprised of 1,300 students. Very few are on free lunch. There is a very small minority population and we have a large, active parent body. Our school is high performing and the majority of students are above grade level.

These students have learned using developmentally appropriate instructional techniques such as Base 10 blocks, Unifix Cubes, Cuisenaire Rods, hexagonal blocks, solid shapes, games, Math Investigations, Hands-On Equations, and so on. They are also taught using scaffolding that follows Singapore Math format and Math CAMMP format, which build concept development through teaching math starting with conceptual understanding, then representational understanding, and finally symbolic algorithms. Despite all of these instructional strategies that are designed to build understanding, my students don't seem to be able to manipulate numbers at the symbolic level. Once they reach the step of learning the traditional algorithm, most of my students rely completely on that pattern in order to solve the problem. The students do not understand that the algorithm is just a series of steps to stay organized when adding the different place value columns.

Rationale

The goal of this unit is to develop mathematical problem solving abilities in my students and to develop a greater understanding of place value. Students will first participate in an activity called "Exploding Dots" to understand the base 10 number system. Next, students will develop different algorithms to add and subtract numbers. Lastly, they will apply their understanding of place value and Exploding Dots to write numbers using the base-two number system, and they will generate their own number system. My goal is not to prepare them for mandatory state testing. It is for students to be able to really practice manipulating numbers and work together to successfully find solutions to challenging problems, so that they like math as well as understand place value. Our school system started a program called Math Investigations this year in which students learn to make sense of their mathematical ideas. This program advocates teaching students that they are able to have mathematical ideas, concepts that they think of or discover for themselves by solving problems. Students can use and develop these mathematical ideas when thinking about and reasoning through unfamiliar problems. "Students are expected to leave Grade 4 fluent in addition, which means that they have at least one strategy they use fluently and efficiently while having access to an understanding of other possible strategies"[1]. In order for students to have this fluency, which they understand conceptually enough to be able to apply in other situations, I have developed these lessons to have students solve problems where they use their knowledge of place value to assist them in creating a new algorithm for addition and then subtraction.

Students, often through drilled practice, understand how to add numbers with the traditional algorithm. For example, to solve the problem 343 + 276, students would line the numbers up in a column, starting with the ones place, continue adding right to left, regrouping as necessary using annotations at the top of the columns, until finished. Many students become fluent with this method only through strict memorization, as often demonstrated by their lack of understanding of what they are doing and their inability to apply an understanding of this method to new situations. Furthermore, when students fail to develop knowledge conceptually in mathematics, they may become frustrated when trying to solve problems, which may lead to a dislike of math.

Most of the work that I have done, and the lessons that are in *Investigations,* involve working at the concrete and representational stages of mathematics. The part of this unit on backwards numbers is designed for use with students who are ready to work at the symbolic level of addition with up to five digit numbers. This means that they should already have an understanding of magnitude and a mastery of addition at the concrete (i.e. Base 10 Blocks), representational (i.e. pictorial Base 10 Blocks), and transitional levels (i.e. partial sums). This unit should not be used with students that have not reached this point. The backwards number lessons are designed to have students make a direct connection of their work with concrete manipulatives and representational figures to the development of the standard algorithm by having the students develop an original algorithm for addition and subtraction with five digit numbers, starting from the left side.

My rationale for having students approach this problem is supported by the

Singapore Math Framework, which includes Metacognition as one of five major components in Mathematical Problem Solving.[2] Metacognition is "thinking about thinking". Singapore Math defines Metacognition as "awareness of, and the ability to control one's thinking processes", which involves two parts - 1) Monitoring of one's own thinking, and 2) Self-regulation of learning. The self-regulation aspect demands that students evaluate their progress in solving a problem and make changes when necessary. Metacognition is an essential part in the development of problem-solving abilities. Therefore, as students work through the problem of developing a different algorithm for adding progressively larger numbers, they will need to monitor their own thinking, as they make sure their connections to what they understand are accurate, and adjust as necessary the efficacy of their reasoning. Exploding Dots is the other instructional method that I will use to build students understanding of place value and develop problem solving. Exploding dots, as explained below, is a representational method of counting. Once a specified amount of dots are accumulated in a box they explode into the adjacent box. The boxes represent the place value columns. In our Base 10 numbers system dots explode into the next box when a tenth dot is added to a box. There may never be 10 dots in a box just as there may never be 10 ones in the ones column.

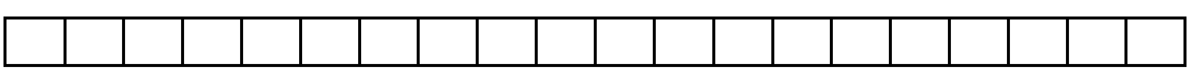
Once students understand how to use Exploding Dots, I will use it to teach them the Base 2 number system (binary code). My rationale for this is that it is that understanding greatly increases when the mind understands an alternate possibility for something. For example, if you have never seen daylight, it is difficult to truly understand night. If you have an animal, a dog, but have never seen another kind of animal, it is difficult to understand what the animal kingdom truly is like or comprised of or to appreciate the value of a dog in relation to other animals. Therefore, when students have the opportunity to count using Base 2, Base 4, or another system they invent, their understanding of the Base 10 system will grow appreciably.

Strategies

Exploding Dots and the Base 10 Number System

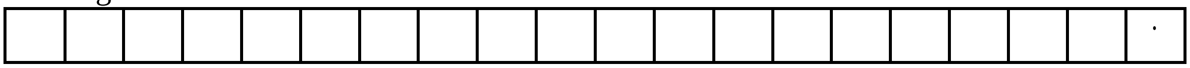
The first part of my unit is for students to learn how to use Exploding Dots with the Base 10 number system.

Step 1) When using Exploding Dots, students have a row of infinite boxes to record dots. These boxes are in a 1 by 8 array:

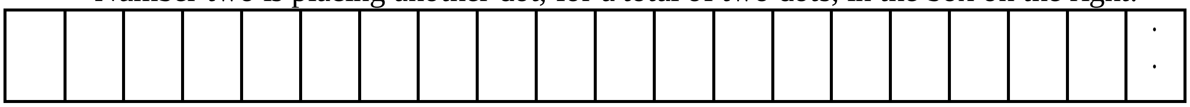


Students only need to draw the number of boxes they are going to use and may add on boxes at any time to their drawings.

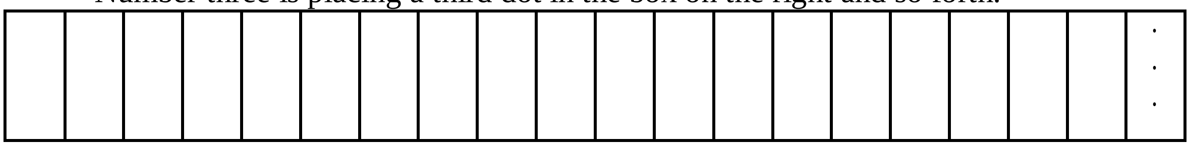
Step 2) Students place dots into the boxes starting with the farthest box on the right side. Basically, this is a number system and we are going to be “counting up” by adding dots on the right side each time we count. Number one is placing a dot in the first box on the right.



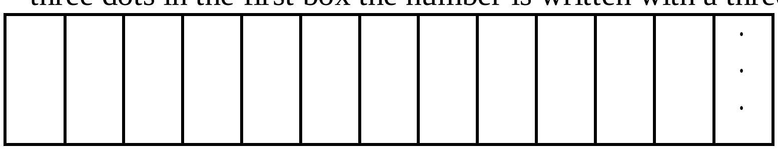
Number two is placing another dot, for a total of two dots, in the box on the right.



Number three is placing a third dot in the box on the right and so forth.



Step 3) Numbers are recorded based on the number of dots in a box. Place value is maintained by the boxes. When working with the first box on the right it is representing the ones column. The next box moving left is the tens column. The third box is the hundreds column, etc. Zeros are used to hold the place value. For example, if there are three dots in the first box the number is written with a three.



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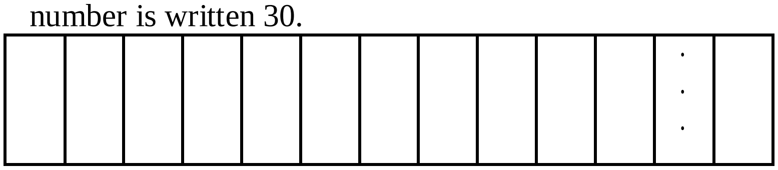
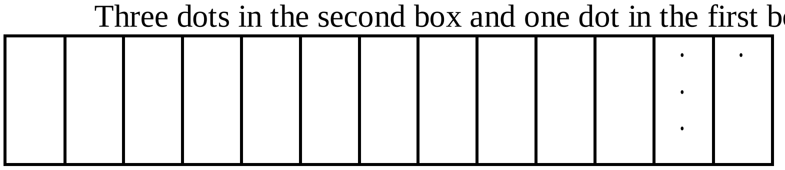
If there are three dots in the second box from the right and none in the first the number is written 30.



30

Three dots in the second box and one dot in the first box is written 31.



31

Step 4) Finally, the exploding part! If there are 10 dots in a box, the dots explode and a new creation is formed – one dot in the next box to the left! The kids love this part!

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These 10 dots explode and create ↓

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There may not be 10 dots in a box. The maximum number of dots there may be is 9. This is our Base 10 number system. There in no digit for the number 10. Our number system is represented by the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. Our Exploding Dots model represents this system.

Treating this model as a machine with rules, we can express our rule as:

1←9

The great part about Exploding Dots is that the machine's rules may change and be a different rule. This will be demonstrated in the lessons following backwards numbers where students will generate the binary system using Exploding Dots and a machine with the rule:

For now, students will draw and record the number to 101 using Exploding Dots. Then I will ask them to un-explode a series of dots to prepare them for the Backwards Numbers lessons. For example, I will ask them to take the number

	· · ·			
	3	0	0	0

and un-explode it one dot at a time, so when one dot is taken away this becomes

	· ·	....	....	....
	2	9	9	9

It may be very difficult for students to realize that to take away one dot from 3000, the thousands, hundreds, and tens columns all have explosions. First, a dot in the thousands columns is exploded to ten dots into the hundreds column. This cannot remain where it is based upon our explosion rule, but we are subtracting and so one of the hundreds dots is removed and changed into ten dots in the tens column. This cannot remain as is, so one dot is removed and put into ten dots in the ones column, where one dot can be removed. This leaves nine dots in each of the thousands, tens, and ones columns.

Backwards Numbers

Students will develop a non-standard algorithm to solve addition and subtraction problems next. This algorithm must be a backwards one or one that starts on the left side and moves right instead of starting on the right side and moving left. In regards to standard algorithms it is interesting to note that the long division algorithm is accomplished backwards as well, with notation occurring below the problem. Three-digit by one-digit division, for example, begins with putting the divisor into the column of the dividend and then proceeding sequentially through the tens and then the ones columns.

As students work through the problem I imagine notation will look similar to the following scenario:

3 4 7 8 2
\+ 9 6 5 8 5
1 ~~2~~ ~~0~~ ~~2~~ 6 7
3 1 3
1 3 1, 3 6 7

Step 1) Ten thousands column added first. Product is 12. Record in its entirety.

Step 2) Next the thousands column is added. Product is 10. Record 0 in the thousands column and add the one to the 2 that is already in the ten thousands column. Do this by crossing off the two and changing it to a three. Record the 3 below the 2.

Step 3) The next step is to add the hundreds column. Product is 12. Record the 2 in the hundreds column and regroup the thousands into the correct column by changing the 0 to a 1, annotating this the same way as in Step 2.

Step 4) Next add the tens column. Product is 16. Record the 6 in the tens column and regroup the 1 into the hundreds column by changing the 2 to a 3.

Step 5) The ones column doesn't require regrouping in this problem, so record the product 7 in the ones column.

Step 7) Write the final answer using the numbers in each column that are not crossed off.

It is worthy to note that it is impossible to have to regroup across two columns, same as in addition, using the regular algorithm as demonstrated here by adding numbers with all 9's.

9 9 9 9 9
\+ 9 9 9 9 9
1 ~~8~~ ~~8~~ ~~8~~ ~~8~~ 8
9 9 9 9_
1 9 9 9 9 8

It is possible that students will develop other viable algorithms and groups of students will present these solutions to the class for discussion. For example, students may choose to annotate the number they are regrouping in some way. Using the previous example this may look like:

3 4 7 8 2
\+ 9 6 5 8 5
1 1 1
1 2 0 2 6 7
3 1 3
1 3 1 3 6 7

In this problem, space is left between the problem and the first line of the answer to record the regrouped number (shown in italics) and then a line is drawn and these numbers are combined.

A student might also choose to circle a number or cross off the old number originally occupying the space before regrouping and record the new number above the old one.

However, the students choose to do it, my criteria for an algorithm are that there is no erasing and their must be consistency in the method. The definition of an algorithm is a set of rules used for calculation or problem solving according to The Oxford American Dictionary of Current English. Therefore, consistency must be present to indicate that there is a set of rules that is being applied and followed. Erasing is not allowed because the work is not apparent because part of it has been erased, making the process difficult if not impossible to follow and analyze.

Students will also create a backwards subtraction algorithm. I envision this algorithm working similarly to the one demonstrated in the following example:

3 1, 6 2 0
\- 1 2, 7 8 5
2

Step 1) In the above problem, the ten thousands column will always produce a one digit answer. This is recorded in the ten thousands column as noted above.

11
3 ~~1~~, 6 2 0
\- 1 2, 7 8 5
~~2~~ 9,
1

Step 2) In the above problem, there is not enough in the thousands column to complete the subtraction process in this column, so the regrouping must occur in the answer section of the ten thousands column since that column has already been subtracted. Taking one group of ten thousand away from the two in this column, leaves a one in the ten thousands column and, when added to the thousands column, leaves 11 thousands to work with. Next, the subtraction can be completed, which gives an answer of 9 in the thousands column.

11 16
3 ~~1~~, 6 2 0
\- 1 2, 7 8 5
~~2~~ ~~9~~, 9
1 8,

Step 3) Similarly to Step 2, there is not enough in the hundreds column to subtract and the regrouping must occur in the answer part of the thousands column. Completing this regrouping puts 16 in the hundreds column and produces and answer of 9 after subtracting.

11 16 12
3 ~~1~~, ~~6~~ ~~2~~ 0
\- 1 2, 7 8 5
~~2~~ ~~9~~, ~~9~~ 4
1 8, 8

Step 4) The tens column is completed using the same procedure for the hundreds and the thousands columns, as shown above.

11 16 12 10
3 ~~1~~, ~~6~~ ~~2~~ ~~0~~
\- 1 2, 7 8 5
~~2~~ ~~9~~, ~~9~~ 4 5
1 8, 8 3
1 8, 8 3 5

Step 5) The final steps are to complete the process in the ones column by regrouping from the answer in the tens column and recording the answer, 18,835. An interesting thing occurs when applying this new method to subtracting across zeros or a middle zero.

With the traditional subtraction algorithm, there ends up being nines in the middle from regrouping twice in each column in order to get the one group of ten thousand moved across the thousands, hundreds, and tens columns, into the ones column. For backwards numbers this does not occur and there are 10’s in each column at the top.

		10	10	10	10
	3	~~0~~,	~~0~~	~~0~~	~~0~~
-	1	2,	7	8	5
	~~2~~	~~8~~,	~~3~~	~~2~~	5
	1	7,	2	1	
	1	7,	2	1	5

When teaching students to tackle this problem, scaffolding may occur through uses two digit plus two digit numbers without regrouping to start followed by problems requiring regrouping and then working up to five digit plus five digit with and without regrouping. After addition, subtraction may be scaffolded using progressively larger numbers, with and without regrouping, followed by subtraction across zeros and across middle zeros.

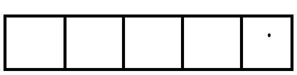
Development of Another Number System

The final step of the unit is to have students comprehend a base-2 (binary) number system or binary system. Students will use Exploding Dots to learn about the binary system (base 2) in a representational way and then they will convert these pictures into numbers using the digits 0 and 1. Afterwards, students will develop their own number system using exploding dots and have classmates create numbers using the number system they developed. All work will be done in small groups to promote learning through social transmission and to add to their enjoyment of solving the problems.

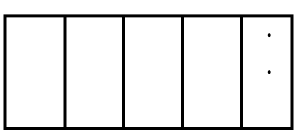
The rule for the binary system in Exploding Dots is

1← 2

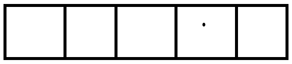
Therefore the first number is 1



The second number has an explosion because two dots may not stay in the same box without exploding.



becomes ↓



This second number is 10 because there is 1 dot in the second place value and 0 holds the place on the first column.

The third number is 11 because a 1 is added to the box on the right and the forth number is 100 because the first box explodes and this puts two dots in the second box, which therefore explodes. The entire system is written in 0’s and 1’s.

After this, students may make there own Exploding Dot machine rules and record their number system as well as try their new rules out on classmates,

Classroom Activities

Lesson on Exploding Dots (Base 10) with Extension Ideas

Title: Exploding Dots

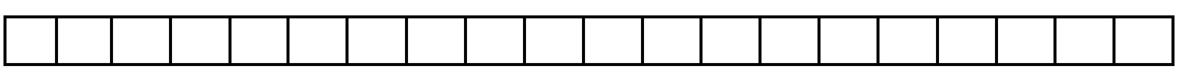
Rationale: Students will understand Exploding Dots as a representation model of our Base 10 number system to build number sense and as a stepping stone to understanding other number systems such as the binary system.

Objective: Students will comprehend that number reorganize after groups of 9 are established.

Focus/Review: What is place value? How do we represent it? To review, we have learned a system of remembering the organization of place value using houses and fences. Every three houses are followed by a fence, which we represent with a comma. Each house is a specific place value. The different houses have family names – the Ones family, the Tens family, the Hundreds family, the Thousands family, etc. Today we are going to learn a new game about place value to help us understand how place value works.

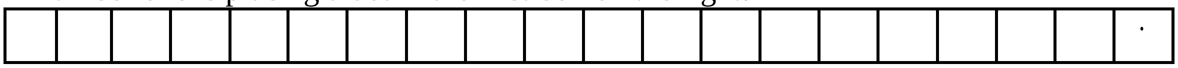
Teacher Input: Exploding Dots is the activity we are going to do today. I will put an array on the board to hold my dots.

Step 1) When using Exploding Dots, you may have as many boxes as you need in your array to record dots.

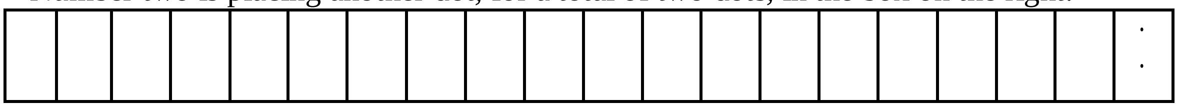


Step 2) Place dots into the boxes starting with the farthest box on the right side. Basically, this is a number system and we are going to be “counting up”.

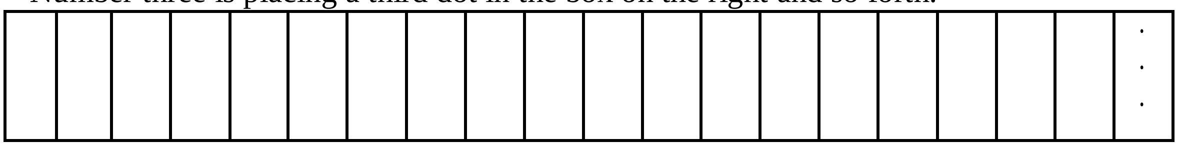
Number one is placing a dot in the first box on the right.



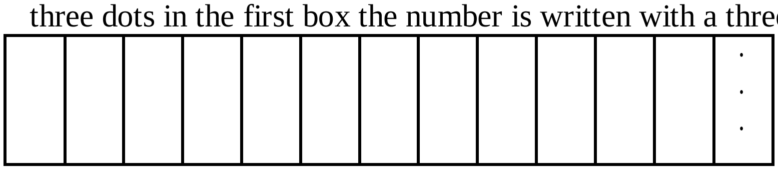
Number two is placing another dot, for a total of two dots, in the box on the right.



Number three is placing a third dot in the box on the right and so forth.



Step 3) Numbers are recorded based on the number of dots in a box. Place value is maintained by the boxes. When working with the first box on the right it is representing the ones column. The next box moving left is the tens column. The third box is the hundreds column, etc. Zeros are used to hold the place value. For example, if there are three dots in the first box the number is written with a three.



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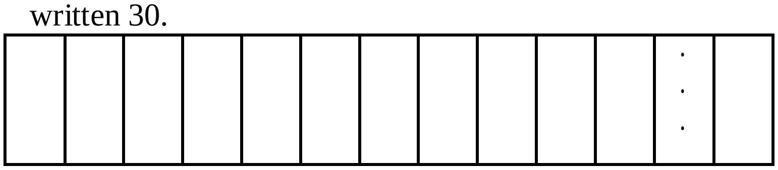

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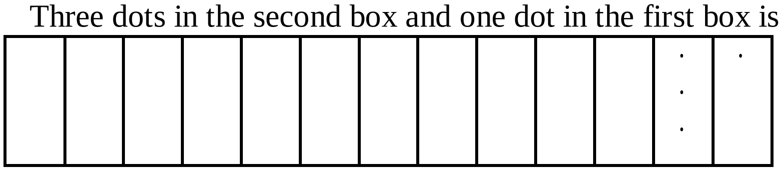

31

Step 4) Finally, the exploding part! If there are 10 dots in a box, the dots explode and a new creation is formed – one dot in the next box to the left!

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These 10 dots explode and create ↓

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There may not be 10 dots in a box. The maximum number of dots there may be is 9.

Guided Practice: Lets do a few numbers together. Here I will ask questions and get the kids to add dots and do explosions on dry erase boards at their desks to check for understanding.

Independent Practice: Next in small groups (I put three in a group. I put students together that have similar understanding of numbers, so that they are equally challenged and not getting the answers from a group member that understands it better than they do) I want you to draw all the numbers to 32 using arrays and dots on chart paper.

Closing: Review their work and have students discuss the process of creating the numbers. Explain that we could write what we did as a rule for exploding the dots. How might we express that rule. Clarify that we will write our rule as:

1←10

Lesson on Backwards Numbers with Extension Ideas

Title: Backwards Numbers

Rationale: Students should understand number well enough to change the standard method of adding numbers and create a new way by starting on the left side of the equation.

Objective: Students will add up to five digit numbers with five digit numbers by using a non-standard method, demonstrating their deep understanding of place value and sense of numeration.

Focus/Review: This lesson says that in order to add numbers you must start on the right side. Is that how we usually add numbers? Is it possible to start on the left side? If so, how would you do it?

Teacher Input: Today we are going to see if we can add numbers starting on the left side. Let's first discuss the steps of how to add problems when we start on the right side. Brainstorm with class the steps to do a three digit by three digit addition problem. They should come up with something along these lines: 1- Put numbers in a column and line up the place values, 2- Add the digits in the ones column, 3- Record the ones below in the ones column, 4- Record tens (if any) at the top of the tens column, etc.

First, I want you and your group of three to write all the steps in order for adding five digit numbers. Then, bring it to me to check. Next, I want you to see if you can develop steps to solve the same equation by starting on the left side. You may have to record the numbers in a different way, which is expected.

Guided Practice: Work with a partner and write the steps of adding five digit numbers for me to check.

Independent Practice: In groups of three students, write an algorithm or steps to solve a five digit by five digit addition problem by starting on the left side. Be prepared to explain your results to the class.

Closing: Numbers may be added backwards. Algorithms for addition are a way of staying organized while adding the different place values. There are more ways than one of working through the problem and organizing the steps. It is important to remember to keep track of your process by annotating each step you take to add the numbers. As long as you do this, you may add numbers any way you would like to – not just from the right side.

Extension: Subtract lesson using similar format. How would this apply to multiplication or division algorithms?

Lesson on Exploding Dots (Base2)

Title: Exploding Dots

Rationale: Students will understand Exploding Dots as a representation model of another number system, Base 2, to build number sense, to begin to build an understanding of the meaning of our own Base 10 number system, that there are different “languages” in mathematics, and to appreciate the beauty and order of a system.

Objective: Students will use number sense to comprehend that place value can be used to organize a number system.

Focus/Review: What is place value? How do we represent it? To review, we have learned a system of remembering the organization of place value using houses and fences. Every three houses are followed by a fence, which we represent with a comma. Each house is a specific place value. The different houses have family names – the Ones family, the Tens family, the Hundreds family, the Thousands family, etc. Today we are going to learn a different system of organizing numbers using place value. It is called the binary system and is used for different things, but most people know it as a system used in computer engineering.

Teacher Input: Do all languages use the same alphabet? If a group of aliens on another planet developed a way to count what do you think the probability is that it would work or look exactly like our number system? Our number system is our invention, just like the words in our language and how those words work together. To illustrate this, I am going to show you the Base 2 number system, as opposed to our Base 10 number system, using Exploding Dots.

Step 1) Remember, when using Exploding Dots, you may have as many boxes as you need in your array to record dots.

Step 2) Place dots into the boxes starting with the farthest box on the right side. Basically, this is a number system and we are going to be "counting up". Place dot after dot in the box on the right side.

Number one is placing a dot in the first box on the right.

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Number two is placing another dot in the box on the right.

																			· ·

Step 3) Numbers are recorded based on the number of dots in a box. Place value is maintained by the boxes. When working with the first box on the right it is representing the ones column. The next box moving left is the tens column. The third box is the hundreds column, etc. Zeros are used to hold the place value. For example, this number would be written 10111. Every box with one dot is represented with the digit 1 and zeros are used to hold the place of empty boxes.

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Step 4) Finally, the exploding part! If there are 2 dots in a box, the dots explode and a new creation is formed – one dot in the next box to the left!

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These 2 dots explode and create ↓

			·	

There may not be 2 dots in a box. The maximum number of dots there may be in a box is 1.

Guided Practice: Lets do a few numbers together. Here I will ask questions and get the kids to add dots and do explosions on dry erase boards at their desks to check for understanding.

Independent Practice: Next in small groups (I put three in a group. I put students together that have similar understanding of numbers, so that they are equally challenged and not getting the answers from a group member that understands it better than they do) I want you to draw all the numbers to 32 using arrays and dots on chart paper.

Closing: Review their work and have students discuss the process of creating the numbers. Explain that we could write what we did as a rule for exploding the dots. How might we express that rule. Clarify that we will write our rule as:

1←2

Extension: I will ask students to write all the number to 32 and the rule for the Base 4 number system and then I will ask them to develop their own number system and use it to write the numbers and arrays to 32.

At this point it is important to discuss what the number 32 represents. When students use the Base 2 number systems they have an easier time continuing the numbers until they have 32 representations. For the extension I will ask students to write in the Base 4 number system. This system has the digits 3 and 2 in it and students will stop when they reach the number 32. However, 32 in the Base 4 system does not represent 32 in the Base 10 system. The best way I have figured out to get them to understand this is to count block, paperclips, or some other manipulative. When they reach the number 32 in the Base 4 system they will not have 32 blocks in their hand. Ask them to explain this and then have them count objects using their new number system.

Notes

[1]This quote is located in *Investigations in Numbers, Data, and Space*, Unit 5, page 11.

[2]This quote is located in *The Singapore Model Method for Learning Mathematics*, p 11.

Resources

A Collection of Math Lessons, 3-5 (Burns, Marilyn., Math Solutions and Publications: 1986, Sausalito). A book full of lesson plans for upper elementary students that build concept development.

The Heart of Mathematics An invitation to effective thinking (Burger, B. and Starbird, M. Key College Publishing: 2005, Emeryville). This textbook focuses on making the important ideas of mathematics accessible to non-mathematicians. It poses intriguing questions and lessons for thinking about math.

Investigations in Number, Data, and Space (NSF, Pearson, TERC. Pearson Education: 2008, Cambridge). This is part of an elementary school curriculum that is designed for teaching mathematics to fourth graders using activities that have students engaged in making sense of mathematical ideas.

Math CAMMP (Piel, J. and Green, M.: University of North Carolina Charlotte, 2003, Charlotte). This is a course pack that explains the CAMMP system of teaching mathematics: Comprehensively Applied Manipulative Mathematics Program. This is a nine step program that builds concept development in mathematics by introducing concepts with concrete manipulatives, then representational methods, followed by the introduction of the standard algorithm. It explains the environment needed by excellent math instruction and ways to assist students who are struggling.

Place Value (Howe, Roger., Yale University, Department of Mathematics: 2007, New Haven). An Essay which outlines ways to make the study of place value more concrete.

The Singapore Model Method for Learning Mathematics (Singapore Ministry of Education:2009, Singapore). This is a book explaining the Singapore Model Method, which is a teaching strategy for mathematics that starts with concrete methods, then representational, and then symbolic. The method focuses on problem solving while emphasizing conceptual understanding.

Starting Off Right in Arithmetic (Howe, Roger: Yale University, Department of Mathematics: 2009, New Haven). An essay outlining the basic learning sequence for early elementary school, which outlines the conceptual foundation students need to be successful in math.