Tessellations and Origami: More than just pretty patterns and folds
The Bridge between Math and Art through the study of Polyhedra

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Introduction

Math and Art, friends or enemies. Most people would not put these two words together in a sentence, much less try to create a lesson or unit that can be just as exciting to experience as Art on its own.

As an artist and art educator, I enjoy the subjective nature of art, as do many of my students. They don't feel that they have to get an answer right, or listen to someone else's opinions, because art is an expression. It is what they want it to be. However, a negative aspect to that is they don't take it seriously and think of it as a "break" from their normal school education. As an educator, I am looking for solutions to stop "breaks" in learning from happening. I believe that art is the bridge that can connect other disciplines and make learning more meaningful, helping students to better retain the information that they have learned outside the art room. My hope is that this concept for learning will help fuel their desire to want to learn more in art and in life.

I know all of us love Math, right? Some of us though, might think of it as overwhelming or tedious. Well, I have had those same thoughts; however I do enjoy finding solutions, and having a definite answer to my question, which doesn't always happen in the art discipline. Solving problems from an equation, a hypothesis, or a personal goal can help build a student's confidence. Students enjoy seeing their projects progress and come to fruition. They also like to solve puzzles. So why should the concept of Art and Math together be so puzzling? Art and Math may seem like polar opposites, but there are in fact many similarities between them.

In art, the use of repetition, pattern, unity, and scale parallel the very same principles found in mathematics. Artwork can consist of tilings, tessellations, and three-dimensional forms of polyhedra in combination with more traditional elements and principles of design to create unique forms of art that stimulate both the left and right side of the brain.

Through this unit, I plan to engage my students through this exploration of math and art with hands-on activities, graphic technology programs, and expressive artwork that make connections from what they learn in mathematics, to the artwork they study in my classroom.

School Information

At our school, Torrence Creek Elementary, we have a large population of high performing students. We also have a large amount of students enrolled at our school, approximately 1300. Because of the size of our school, I am one of two art teachers here who teach Kindergarten through Fifth Grade. For this unit, I want to focus on challenging our fifth grade students. I teach three of the fifth grade classes this year, for 40 minute periods, once a week. This unit contains a variety of math related art projects and will stretch for a total of 6 weeks and include time in a computer lab.

As an educator, it is my responsibility to study my students' abilities and design my lessons in such a way that pushes them to develop upon the skill sets that they already have. Many of the students at our school are advanced, so this requires that I find ways where they can grow while not pushing others too hard, leaving them discouraged and falling behind. I believe the most successful way to teach a child is to get them excited about their own potential, so they learn to teach themselves through self discipline. My goal is to initiate that spark within each of the students that helps them not only to want to learn, but specifically for the purposes of this unit, to find the connections between art in my classroom and other disciplines. This will help them to merge their creative and analytical critical thinking skills.

Rationale

The objective of this unit is to stimulate the students through the integration of mathematics and art, which will help to expand their own learning potential inside and outside the classroom. During the beginning activities with manipulatives, students will be able to identify various symmetries and properties that make up tilings and tessellations, such as reflections, rotations, and glide-reflections. They will review transformations of the plane figures through experimentation on the M.C.Escher's website, and various computer java programs. Students will become familiar with computer techniques and how to create and represent polygons and their designs. Through this process, the students will connect what they have learned in their regular classrooms with mathematics. They will be able to recognize and use geometric properties of plane figures. They will accurately represent two-dimensional and three-dimensional figures, predict the transformations of plane figures, test conjectures and problem solve with polygons and their characteristics.

Fifth Graders will also be able to identify and analyze the characteristics artwork done by the artist, M.C. Escher, who used a variety of mathematical principles when creating his artwork. Students will use the four levels of art criticism; describe, analyze, interpret, and judge to understand how Escher used line, color, pattern, and form within his work. The critique will focus on the artwork's relationship to the design principles, emphasis, movement, repetition, space, balance, value, and unity.

Students will discover other artists, specifically sculptors, who use designs with polygons in their artwork in the form of polyhedra. Students will then use computer graphics to explore mathematical principles found within polyhedra, and break the forms down into their simplest form to better understand how they're constructed. They will analyze their models using Euler's formula and show symmetries, such as two and three-fold rotational symmetry, dualities and discuss colorability. Students will be able to classify these based on their symmetries and characteristics.

After studying these concepts, students will use pattern, repetition, and unity when creating a group sculpture. Students will learn about the art of Origami, and how to fold platonic solids (regular polyhedra) from origami paper. As small groups, students will then assemble their constructions together to create a larger semi-regular polyhedral sculpture or an abstract sculpture composed of platonic solids.

Background

Tessellations and Tilings

A tiling is a repeated pattern on a plane. It is an arrangement of objects completely covering a plane such that any two tiles, either, share a common vertex, intersect along a pair of edges, or do not intersect at all. Tiles are polygons, the most common being shapes of triangles, squares, and hexagons. There are 17 types of tilings of the planes altogether (see Figure 1). Tilings are "regular" if they are created with regular polygons. There are a total of 11 regular tilings called Archimedean tilings, made with at least two types of regular polygons.

(Fig. 1) 17 Regular Tilings

Tilings and tessellations are made up of symmetries. *A figure in the plane is symmetric if you could pick up a copy of it, perform a rigid motion, and set it back down on the original figure so that it exactly matches up again.* One of the first things to notice about symmetry is that there are several different kinds. One way is to just translate it. Another is to rotate it. Yet another is to turn it over. As a consequence, there are different kinds of symmetry. There are actually four distinct kinds of symmetry, corresponding to four basic ways of moving a tile around in the plane: translations, rotations, reflections, and glide reflections as seen in chart below and Figures 3 through 5.

Translation

Reflection

Rotation

Glide Reflection

A translation is a shape that is simply translated, or slid, across the paper and drawn again in another place.

The translation shows the geometric shape in the same alignment as the original; it does not turn or flip.

A reflection is a shape that has been flipped. Most commonly flipped directly to the left or right (over a "y" axis) or flipped to the top or bottom (over an "x" axis), reflections can also be done at an angle.

If a reflection has been done correctly, you can draw an imaginary line right through the middle, and the two parts will be symmetrical "mirror" images. To reflect a shape across an axis is to plot a special corresponding point for every point in the original shape.

Rotation is spinning the pattern around a point, rotating it. A rotation, or turn, occurs when an object is moved in a circular fashion around a central point which does not move.

A good example of a rotation is one "wing" of a pinwheel which turns around the center point. Rotations always have a center, and an angle of rotation.

In glide reflection, reflection and translation are used concurrently much like the following piece by Escher, Horseman.

There is no reflectional symmetry, nor is there rotational symmetry.

(Figure 2) Translation

(Figure 3) Glide-Reflection

(Figure 4) Rotation

(Figure 5) Reflection

The Classification Theorem for Plane Symmetries: Every symmetry of the plane is either a composition of a translation followed by a rotation, or it's a composition of a translation followed by a reflection.

The difference between a tiling and tessellation is that a tessellation is periodic tiling and a tiling by itself can be both periodic or aperiodic.

A regular tessellation is a highly symmetric tiling made up of congruent regular polygons. Only three regular tessellations exist: those made up of equilateral triangles, squares, or hexagons. A semi-regular tessellation uses a variety of regular polygons; there are eight of these. The arrangement of polygons at every vertex point is identical. An edge-to-edge tessellation is even less regular: the only requirement is that adjacent tiles only share full sides, i.e. no tile shares a partial side with any other tile. Other types of tessellations exist, depending on types of figures and types of pattern. There are regular versus irregular, periodic versus aperiodic, symmetric versus asymmetric, and fractal tessellations, as well as other classifications.

Polyhedra

What is polyhedra? Simply stated it is a three-dimensional object constructed from polygons; mostly squares, triangles, hexagons, pentagons, and/or octagons. Each of these polygons have faces, edges, and vertices. The most simple polyhedral are platonic solids: tetrahedron (made with 4 triangles), cube (made with 6 squares), octahedron, (made with 8 triangles), dodecahedron, (made with 12 pentagons), and icosahedrons (made with 20 triangles). Please see Figure 3 for platonic solid characteristics.

The platonic solids are "regular", meaning the arrangement of regular polygons at the vertices are all alike and made up of only one type of polygon. Platonic solids do not have to be depicted as "solid". They can be hollow, skeletal (no faces), or have perforated surfaces. The regular polyhedra always have mirror symmetry: they can be divided into mirror image halves in many different ways, and they have rotational symmetry: they can be rotated without changing their apparent position. Characteristics of Platonic Solids are shown in Figure 6 and 7 below.

Platonic Solids	**Faces**	**Number of Sides of each face**	**Edges**	**Vertices**
Tetrahedron	4	3	6	4
Cube	6	4	12	8
Octahedron	8	3	12	6
Dodecahedron	12	5	30	

20
Icosahedron
20
3
30
12

(Figure 6) Chart of Platonic Solids

(Figure 7) Platonic Solids

Every surface of each platonic solid has an Euler characteristic of 2. This means that if you let F=the number of faces, E=the number of edges, and V=the number of vertices, then F – E+V=2. This number is the same when finding the Euler characteristic of the Archimedean solids (see Figures 8 and 9). These are made when you take a platonic solid and cut the corners or edges, also known as truncating. The Archimedean solids are considered semi-regular because they are made with two or more kinds of regular polygons. There are eleven of these plus two other semi-regular solids. Four types of stellated polyhedra also exist, produced with pentagrams. Polyhedra are found in nature, in art, and in the makeup of humans. One of the first polyhedra buildings was created in 2500 BC in Egypt, the Great Pyramids of Giza. In nature, polyhedra are found in crystals and minerals, honey combs, and plants.

Archimedean Solids
Types of Faces
Number of Faces
Edges
Vertices
Truncated Tetrahedron
4 triangles and 4 hexagons
8
18
12

Cuboctahedron
8 triangles and 6 squares
14
24
12
Truncated octahedron
6 squares and 8 hexagons
14
36
24
Truncated cube
8 triangles and 6 octagons
14
36
24
Rhombicuboctahedron
8 triangles and 18 squares
26
48
24
Truncated cuboctahedron
12 squares, 8 hexagons, and 6 octagons
26
72
48
Snub Cube
32 triangles, and 6 squares
38
60
24
Icosidodecahedron
20 triangles, and 12 pentagons
32
60
30
Truncated icosahedrons
12 pentagons and 20 hexagons

32
90
60
Truncated dodecahedron
20 triangles, 12 decagons (ten-sided polygon)
32
90
60
Rhombicosidodecahedron
20 triangles, 30 squares, and 12 pentagons
62
120
60
Truncated icosidodecahedron
30 squares, 20 hexagons, and 12 decagons
62
180
120
Snub dodecahedron
80 triangles and 12 pentagons
92
150
60

(Figure 8) Chart of Archimedean Solids

(Figure 9) Archimedean Solids

Rotational Symmetry

An object has rotational symmetry when you can rotate it through a certain angle and it still has the same appearance.

The figures below have rotational symmetry:

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   (a.)        (b.)  /      (c.)|______    (d.)
   =====               /\         |   |       /___
                     /  \       |   |     \/  \
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=====    __/____\    __|____|    _\_ /\
           \        |       /
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In Figure a., the parts are related by a rotation around the center by 180 degrees. The figure looks the same twice in a 360-degree rotation. It has two-fold symmetry. In Figure b., it looks the same three times during a 360-degree rotation and is said to have three-fold symmetry. In Figure c., it has four-fold symmetry and Figure d. has six-fold symmetry.

Duality

For every polyhedron there exists a dual polyhedron. Starting with any regular polyhedron, the dual can be constructed in the following manner: (1) Place a point in the center of each face of the original polyhedron; (2) Connect each new point with the new points of its neighboring faces; (3) Erase the original polyhedron.

For example, starting with a cube, you create six points in the centers of the six faces, connect each new point to its four neighbors, create 12 edges, and erase the cube to find the result is an octahedron, consisting of eight triangular faces. So the dual of the cube is the octahedron as shown in Figure 10.

(Figure 10. Dual of a cube) (Figure 11. Dual of an icosahedron)

This is an operation "of order 2" meaning that taking the dual of the dual of *x* gives back the original *x*. For example, take the dual of the octahedron and see that it is a cube. The six 4-sided faces of the cube transform into the six corners of the octahedron, with 4 faces meeting at each. The eight 3-sided faces of the octahedron transform into the eight corners of the cube with 3 faces meeting at each. Also observe that the total number of edges remains unchanged, as each original edge crosses exactly one new edge. The cube and octahedron each have twelve edges. The dual of the icosahedron is the dodecahedron and vice versa as shown in Figure 11. The twenty 3-sided faces and twelve 5-way corners of the icosahedron correspond to the twenty 3-way corners and twelve 5-sided faces of the dodecahedron. Each has thirty edges. The dual to the tetrahedron is another tetrahedron facing in the opposite directions. All the above polyhedra are regular polyhedra, which have the special property that their duals are also regular polyhedra. They also all have the same axes of symmetry. However, when you take the duals of the Archimedean solids, you get a new class of solids called Archimedean duals.

Many teachers have developed hands-on lessons that use origami to make math come to life for their students. Topics taught in this way range across the entire curriculum: problem solving; precise use of mathematical terminology; ratios, fractions, and percents; angles; area and volume; congruence; tessellations; combinatorics; properties of parallel lines; products and factors; conic sections; Euler's formula; logic; and proofs. Origami also abounds with accessible open problems that give students a chance to contribute original ideas.

For this unit, we will only focus on Modular origami. Modular origami, or unit origami, is a paperfolding technique which uses multiple sheets of paper to create a larger and more complex structure than would be possible using single-piece origami techniques. Each individual sheet of paper is folded into a module, or unit, and then modules are assembled into an integrated flat shape or three-dimensional structure by inserting flaps into pockets created by the folding process. These insertions create tension or friction that holds the model together.
(Figure 12) Modular origami made up of Sonobe units

Modular origami, as seen in Figure 12 above, can be classified as a sub-set of multi-piece origami, since the rule of restriction to one sheet of paper is abandoned. However, all the other rules of origami still apply, so the use of glue, thread, or any other fastening that is not a part of the sheet of paper is not generally acceptable in modular origami. The additional restrictions that distinguish modular origami from other forms of multi-piece origami are using many identical copies of any folded unit, and linking them together in a symmetrical or repeating fashion to complete the model.

(Figure 13) Example of Kusudama

Modular origami forms may be flat or three-dimensional. Flat forms are usually polygons (sometimes known as coasters), stars, rotors, and rings. Three-dimensional forms tend to be regular polyhedra or tessellations of simple polyhedra.

The first historical evidence for a modular origami design comes from a Japanese book by Hayato Ohoka published in 1734 called *Ranma Zushiki*. It contains a print that shows a group of traditional origami models, one of which is a modular cube. The cube is pictured twice (from slightly different angles) and is a tamatebako, or a 'magic treasure chest'.

Most traditional designs are however single-piece and the possibilities inherent in the modular origami idea were not explored further until the 1960s when the technique was re-invented by Robert Neale in the USA and later by Mitsonobu Sonobe in Japan. Since then the modular origami technique has been popularized and developed extensively, and now there have been thousands of designs developed in this repertoire.

There are several other traditional Japanese modular origami designs, including balls of folded paper flowers known as kusudama, or medicine balls shown in Figure 13 above. These designs are not integrated and are commonly strung together with thread. The term kusudama is sometimes, rather inaccurately, used to describe any three-dimensional modular origami structure resembling a ball.

Connections between Tessellations and 3-D Polyhedra Forms

Surface Design on Solids

On the surface of a 3-d object, repetitions of a design (a tessellation) can be realized with only a finite number of figures. The solid's pattern has no beginning or end. On the surface of a solid, the centers of rotation of the flat design become the centers of rotation of the solid. In Figure 14 below, you can see an example of a surface design on a tetrahedron.

A rotation of the tetrahedron on an axis is called three-fold rotational symmetry (piercing through the top vertex with an axis). The tetrahedron also has two-fold rotational symmetry when the axis pierces the midpoints of two nonadjacent edges.

In an octahedron design, each equilateral triangle containing interlocking halves of three different motifs make up its face. It needed 12 motifs (4 of each kind).
Four colors are sufficient to color a map of any plane design, but aesthetically it is more pleasing if they emphasize the symmetry of the design.

(Figure 14) Surface design on a Tetrahedron

When a pattern is folded to form a solid, different portions of the design are brought together. Most of the time, it causes the adjacent designs on the solid to have the same color. In order for some of these designs to work, a forced rearrangement of color had to be applied to the surface.

Each patterned solid can be colored evenly, meaning that each different color is used the same number of times.

Classroom Strategies

Activity 1

The unit will begin with student experimentation and exploration. This act of constructing and experimenting raises questions for them about the relationship between math and art, and is something tangible for them to get excited and intrigued about. The students will use manipulatives, such as Tangrams or Geosnaps, or other similar models to create tilings and tessellations. They will take turns experimenting and constructing using these and various internet websites. Please see a list of websites in the appendix for your students to search. Students will visit these sites and use polygons to create their designs. They will choose one to print, then describe and analyze their design and try to rationalize why they "work".

- What polygons are included in your design?
- Are they regular or irregular? How do you know?
- How many colors make up your design? How else could you color it?
- Why do you think your design works? What would happen if you changed one of the polygons?
- Draw an example of a design that does not cover the plane, one that doesn't work.

Activity 2

Students will create constructions of polyhedra on their own with simple materials. Students will be provided with a list of websites such as the National Library of Virtual Manipulatives and computer programs like Geometria and JavaGami. The fifth grade students will be given a worksheet to document their findings about the models.

1) How would you describe the characteristics of a polyhedron?
2) What are some common polyhedrons?
3) What is the relationship between the number of vertices, faces, and edges of any face of a polyhedron?
4) How would you describe the faces of a tetrahedron? Hexahedron?

Octahedron?

5) How many faces share one edge in each of your polyhedrons?

6) What seems to be the least number of edges that meet at each vertex?

7) Which of the polyhedrons that you have built represent Platonic Solids? How do you know?

- How are the Platonic Solids different from the other polyhedrons? How are they alike?

9) Are there any similarities between your findings with tilings and tessellations and your findings with polyhedra?

I will then review the definition of a polyhedron, and check for understanding of their properties. The class will share examples of their findings and connections between polyhedra and the real-world based on their experience and knowledge. I will also share information about Euler's formula and its how it relates to the polyhedra. Students will then have time to apply Euler's formula to several problems and discover for themselves how it works. The relationship among the faces (F), vertices (V), and edges (E) for a convex polyhedron, for example platonic solids and Archimedean solids, can be expressed as **F –E+V=2.**

Activity 3

During the following art class, the students will then be shown examples of various artworks that consist of tessellations and polyhedra. The class will be divided into groups and participate in discussions and critiques of these artworks to better comprehend their purpose and meaning. The class will discuss the symmetries found in the artworks of M. C. Escher and George Hart, as well as others, and critique them based on the four levels of art criticism as mentioned earlier.

Description:

- **What kinds of things do you see?**
- What words would you use to describe this work?
- How would you describe the lines in this work? The shapes? The colors? What is its subject matter?
- How would you describe this work to a person who could not see it?
- What does this work remind you of?
- What things do you recognize? What things seem new to you?

- What interests you most about this work of art?
- What tools, materials, or processes did the art maker use?

Analyze it:

- What elements did the maker choose and how did the maker organize the elements?
- How is the work deal with space?
- Did the artist choose a color scheme? If so what type? What color is used the most in this painting?
- What do you think is the most important part of this work?
- How do you think the artist made this work?
- What questions would you ask the artist about this work, if he were here?

Interpret it:

- What title would you give to this painting? What made you decide on that title?
- What other titles could we give it?
- What do you think is going on in this work? How did you arrive at that idea?
- What do you think this work is about? How did you come up that idea?
- Pretend you are inside this work. What does it feel like?
- Why do you suppose the artist made this work? What makes you think that?

Evaluate it.

- What do you think is good about this work? What is not so good?
- Do you think the person who created this do a good or bad job? What makes you think so?
- Why do you think other people should see this work of art?
- What do you think other people would say about this work? Why do you think that?
- What would you do with this work if you owned it?
- What do you think is worth remembering about this work?

We will end this critique with a presentation of the artworks. A representative will be an “Art Dealer” representing the artist, and I will be the “Art Curator” for the “Class Gallery”. The art dealer will provide a detailed synopsis of the work to give the curator enough information to determine if the work deserves placement in the gallery.

Activity 4

Printed Translation Tessellation:

Materials needed:
12 x 12 white paper
3 x 3 cardboard or chipboard piece
Masking tape
Scissors
Tempura paints (variety of colors)
Trays for paint
Brushes
(newspaper to protect tables) optional

Procedure:

Students will create a design using be tested on their knowledge of tessellation symmetry when creating a simple printed tessellation using cardboard and paint on paper. Students will first fold their 12 x 12 paper 4 times so they create a 3 x3 grid across the paper. They will create a tessellating shape with cardboard by cutting the bottom and sliding it up to the top and taping it down. Then they will cut out a shape from the left side and slide it to the right and tape down. (You want to make sure whatever amount is taken away that it is replaced on the opposite side) Students will then use that shape as a stamp. They will choose one color to begin painting with and will use a brush to paint the color on top of the cardboard shape. They will line the shape up to the bottom right corner of the first square and print it. They will skip every other one to make a checker board pattern with the color. They will repaint as needed. Students will repeat this process with their chosen second color. (They will wipe off any of the first color and print in-between the areas they skipped). Once dry, students can use sharpie markers to add details into their design. They can make their shapes look like a recognizable object or they can add abstract designs onto their design.

Activity 5

Fifth graders, at the next art period meeting, will complete a warm-up activity where they will color in Escher's fish cube design using the following rules.

(1) Each fish is one color.
(2) No two adjacent fish have the same color.
(3) Exactly four colors are used.
(4) Each color is used to color exactly three of the twelve fish.
(it is possible to "three-color" the design)

Taking what they have learned, we will discuss colorability of a platonic solid. Students will then construct a platonic solid from its net (printed from the computer) and will create a simple design that emphasizes the construction and relationship of the faces and assemble it together. I will be monitoring their work and asking questions to check and clarify their understanding of the investigations. Students will use problem solving skills when finding answers to these questions.

(Figure 15) Example of a net

1) How few colors are required to color the map of my design?
2) How many different colors or color combinations can I use?

Activity 6

Modular Origami:

For this project you will need lots of origami paper (several 100pks of a variety of colors) and glue for assemblage. Students will learn how to fold a platonic solid with Origami paper using examples from modular origami instructions. (You can do more complicated solids for older students). They will then assemble them together with a group to create a large polyhedra sculpture. It can be a semi-regular polyhedra or an abstraction made up of polyhedra forms. You can find many instructions online or in books. I used "Unit Polyhedron Origami" by Tomoko Fuse for my square and equilateral triangle units.

For my classroom, I am assigning groups of four, either square or triangle units to make. At each table I will have the step-by-step instructions written, along with an origami paper folded showing that step done. This helps out a lot when you have a large class and they need help to folding the paper. Students in that group will create as many units and joints they need to create their sculpture. For example, if they are creating a sculpture with regular tetrahedron, they will need 4 triangular flat units and 6 joints for each. I have them store their pieces in shoeboxes as they work. They can also take them home and create some as well. Once they fold the first couple, they become experts. Once they fold several solids, they can begin brainstorming ideas how they can connect them together for other polyhedra forms or abstract sculptures.

Activity 7

For the last activity, students will review what they've learned about symmetries and polyhedra and will draw a diagram of their final sculpture. They may show a single polyhedral component and its dual or the entire sculpture's dual. They will explain its characteristics and symmetries, particularly rotational symmetries. They will also reflect upon their process, individual work, and they will analyze their group's work based on the four levels of art criticism. Each student's work and process will be assessed and then displayed in the hallways.

Resources

Barnette, David. *Map Coloring Polyhedra and the Four Color Problem (Dolciani Mathematical Expositions)*. Unknown: Mathematical Assn Of Amer, 1984.
This book is a good resource to explain colorability of polyhedra and other maps.
Escher, M. C., and John E. Brigham. *M.C. Escher: the graphic work*. Berlin: Taco Verlagsgesellschaft und Agentur mbH, 1989. This resource contains several visuals and coordinating information for Escher's work.
Farmer, David W.. *Groups and Symmetry: A Guide to Discovering Mathematics (Mathematical World, Vol. 5)*. Providence: American Mathematical Society, 1995. This book was used as a resource for explaining symmetries within wallpaper patterns.
Fuse, Tomoko. *Unit polyhedoron origami* . Tokyo, Japan: Japan Publications ;, 2006. This book was used for instructions of modular origami folds and forms.
Hilton, Peter and Jean Pederson. *Build your own Polyhedra*. Addison-Wesley Publishing Company, 1988. This resource has various instructions and example of polyhedral forms. It shows how to construct a polyhedra from it's net.
Holden, Alan. *Shapes, Space, and Symmetry*. New Ed ed. New York: Dover Publications, 1991. This book explains more in depth the characteristics of platonic solids, Archimedean solids, and various other polyhedra. It has great visuals and explains symmetries of the three-dimensional forms.

Hull, Thomas. *Project Origami: Activities for Exploring Mathematics*. Natick: A K Peters, Ltd., 2006. This resource shows various origami and mathematical connections. The book has examples of folding equilateral triangles and several polyhedral forms.
Kalajdzievski, Sasho. *Math and Art: An Introduction to Visual Mathematics*. 1 ed. Boca Raton: Chapman & Hall/Crc, 2008. This resource was used for mathematical and visual information concerning tilings, tessellations, and polyhedra.
Schattscheider, Doris. *M.C. Escher Kaleidocycles*. Rohnert Park: Pomegranate Artbooks, 1977. This is a great resource for examples of Escher's work, explanations, and actual paper models of his designs on platonic solids and other forms.
Senechal, Marjorie. *Shaping Space: A Polyhedral Approach (Design Science Collection)*. Basel, Switzerland: Birkhauser, 1988. This resource shows examples of polyhedra.
Singer, David A.. *Geometry: Plane and Fancy (Undergraduate Texts in Mathematics)*. 1 ed. New York: Springer, 1998. This resource shows the mathematics behind tiling the sphere with regular and semi-regular polyhedra.

Web Resources

"Archimedean Solid -- from Wolfram MathWorld." Wolfram MathWorld: The Web's Most Extensive Mathematics Resource. http://mathworld.wolfram.com/ArchimedeanSolid.html (accessed November 29, 2010). Image of Archimedean Solids used in Figure 9 of unit.

"BBC - KS3 Bitesize: Maths - 3D shapes - Nets." BBC - Homepage. http://www.bbc.co.uk/schools/ks3bitesize/maths/shape_space/3d_shapes/revise3.shtml (accessed November 29, 2010). Image of a net of a cube used in Figure 15 of unit.

"Duality." George W. Hart --- Index. http://www.georgehart.com/virtual-polyhedra/duality.html (accessed November 29, 2010). Images of duals of a cube and icosahedron used in Figure 10 and 11 of unit.

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