The Golden Ratio: Making Math Beautiful

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Rationale

The Golden Ratio is a fascinating topic. It is found abundantly in nature and wondrously in the design and makeup of the human body. It has been used for centuries in the construction of architectural masterpieces and by the great artists, who, being able to see its beauty, used it in their designs and compositions. The Golden Ratio is everywhere; mathematics is all around us.

For many of my students math is perceived as a dry subject; lifeless and, as some of my students would say, pointless. “Why do we need this?” “When am I ever going to use this in life?” These are just a couple of the questions that we as math teachers are combated with in the classroom. The Golden Ratio, in its prolific abundance in the world in which we all inhabit, is a mathematical concept that has the ability to transcend the barrage of negativity we often receive from our students. The Golden Ratio is a topic that is broad and deep. There are a diverse number of directions, paths and tangents to which the study of this beautiful concept could take us. It will provide a means to engage students in the world in which they live, but also, hopefully, spark a mathematical interest in those students we find hardest to reach.

Luca Pacioli is quoted as saying, “Without mathematics there is no art.” [1] Throughout this unit we will investigate the math that defines beauty. We will look at the use of the Golden Ratio in ancient, classical, and modern architecture, and in artwork and photography. Finally, we will discover the Golden Ratio as it appears in the human body. Dean Schlicter said, "Go down deep enough into anything and you will find mathematics." [2] It is my desire to inspire my students to dig down deep enough to find the joy and beauty of math in everything.

Objective

I teach in a middle school with a population of approximately 1400 students. The school is located just outside of Charlotte, North Carolina. It is part of the Charlotte-

Mecklenburg school district. My school is part of the Northeast Learning Community. The school is in an affluent area and populated mostly by students whose parents are well-educated and have high-paying jobs. Our community is also growing rapidly, with many newcomers moving from Northern states such as New York, Michigan, and Ohio. While the majority of our student population is white (78%), we have a growing number of minority groups. The makeup of the school's minorities is as follows: 13% African American, 7% Hispanic, and 2% Asian. [3] As stated above, our school is located in an affluent community; even so, 25% of our students receive free or reduced lunch.Six percent of our population is limited in English proficiency. Many of our LEP students come from countries like Mexico, Brazil, Guatemala, Honduras, Germany, and Sweden. Seven percent of our students are identified as having some type of learning disability and receive services from our EC department. [4]

Middle school math is broken into two levels at my school. Standard Plus classes contain students who may be well-below grade level, below grade level, or at grade level. Honors classes contain students who are at, or above grade level. Standard Plus classes teach on grade level objectives. Honors classes teach objectives that are a year in advance, whilst maintaining objectives for the students' current grade level. In eighth grade, that means that Standard Plus classes learn objectives from the 8th grade North Carolina Standard Course of Study (NCSCOS), and Honors classes take Algebra 1 (traditionally a ninth grade course) receiving high school credit upon completion of the End of Course test at the end of the year. Standard Plus classes tend to have more diversity of race and socio-economic status than Honors classes. Students in Honors classes are more likely to be self-motivated. Standard Plus classes, because of their innate diversity, allow for a greater range of learning.

This unit is designed specifically for middle school students. I have included extensions that I plan to use with my eighth grade Algebra students. Students will be exposed to a range of architecture, artwork and photography that incorporates the Golden Ratio in its design. They will be asked to construct Golden Rectangles and Spirals. Through specific artwork (and photographs) students will see that the human body also has inherent Golden Ratios. This will be explored through a measurement activity, where students are allowed to discover Golden Ratios in their own bodies. By the completion of this unit students will have learned about the history of the Golden Ratio, constructed a Golden Rectangle, discovered evidence of its use in architectural structures and paintings, and compared the ratios of their body parts to the Golden Ratio.

Background Information

History

Phidias, a Greek sculptor c. 490 – 439 BC, used the ratio in the design of many of his sculptures for the Parthenon. Later Plato, a Greek philosopher, alluded to the proportional relationship in his work *Timaeus*. [5] However, it was in Euclid's *Elements* that the definition of the Golden Ratio was first written. [6] Euclid explains that "A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the less. [7] He found that this ratio was an irrational number.

Leonardo Fibonacci (c. 1175-1250 AD), wrote about a series of numbers in his book *Liber Abaci*. The series: 1, 1, 2, 3, 5, 8, 13…, continues to increase by adding the previous two numbers in the series. [8] If we let F stand for a Fibonacci number and n stand for the number of the term in the sequence, then $F_n = F_{n-1}+F_{n-2}$. [9] These numbers appear repeatedly in nature. Many types of flowers have a Fibonacci number of petals. "For example, an iris has 3 petals, a primrose 5, a delphinium 8, ragwort 13, an aster 21, daisies 13, 21, or 34, and Michaelmas daisies 55 or 89 petals." Sunflowers also have spirals (clockwise and counterclockwise) at their centers that are Fibonacci numbers: "21 or 34 clockwise and 34 or 55 counterclockwise, respectively." Pine cones and pineapples also have spirals made of Fibonacci numbers. [10] The numbers in Fibonacci's sequence, when taken in pairs of ratios of consecutive terms, approach that of the Golden Ratio, which is 1.618… [11]

Figure 1: Sunflower, from M. Bourne, "The Math Behind the Beauty," http://www.intmath.com/Numbers/mathOfBeauty.php (accessed October 4, 2010).

The "extreme and mean ratio," as it was called, was not referred to as "golden" until the early 16th century AD. In his book, *La Divine Proportione ("The Divine Proportion")*, Luca Pacioli wrote an in-depth study of the Golden Ratio. [12] He wrote, "Like god, the Divine Proportion is always similar to itself." [13] Leonardo Da Vinci contributed several drawings to *La Divine Proportione* and made reference to the "section aurea" (Latin for "golden section"). [14] There was a renewed interest in the "Divine Proportion" during the Renaissance amongst artists, architects, scientists, and mystics. The idea that the use of the Golden Ratio in art and architecture was aesthetically pleasing was widely established during this time. [15]

In 1597, Michael Maestlin, a German professor, in a letter to his former student, Johannes Kepler, wrote that the inverse of the golden ratio is “about 0.6180340.” [16] Kepler went on to prove that the Golden Ratio is the limit of the ratio of consecutive Fibonacci Numbers. [17] He is also quoted as saying, “Geometry has two great treasures: one is the Theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first we may compare to a measure of gold; the second to a precious jewel.” [18]

In the twentieth century, American mathematician, Mark Barr, began using the Greek letter Φ or φ (*phi*, after the Greek sculptor, Phidias) to represent the Golden Ratio, which by this time was also called the Golden Mean, the Golden Section, the Golden Proportion, and the Divine Proportion. [19] In mathematical circles the Greek letter τ (*tau*, from the ancient Greek root meaning “cut”) is also used to refer to the Golden Ratio. [20]

More recently, the Golden Ratio has been instrumental in the discovery of a five-fold symmetry by English mathematician, Roger Penrose, in 1974. The tiles he used to aperiodically tile a plane were formed from pentagons, which are closely linked to Golden Ratios. The patterns of Penrose tilings were also found in quasi-crystals in the 1980’s. [21]

Figure 2: Examples of Penrose tilings, from Toni Beardon, "nrich.maths.org :: Mathematics Enrichment :: Golden Mathematics." http://nrich.maths.org/public/viewer.php?obj_id=2787&part=index&nomenu=1 (accessed November 5, 2010).

Mathematically Speaking

Suppose we have a line segment . The point C on the line divides the segment into a larger part and a smaller part. If the ratio of the length of the whole to the length of the longer segment is proportional to the ratio of the length of the longer segment to the shorter segment , then this point is regarded as the golden cut and the ratios formed are referred to as the Golden Ratio.

Figure 3: from Mario Livio, "The golden ratio and aesthetics | plus.maths.org." http://plus.maths.org/issue22/features/golden/ (accessed November 7, 2010).

=

If we set the length = x, and let the length of the longer segment = 1, then the length of the shorter segment must equal x - 1.

=

We cross-multiply to solve the proportion:

$$x^2 - x = 1$$

We set the equation equal to zero and solve the equation using the quadratic formula:

$$x^2 - x - 1 = 0$$

$$x =$$

$$x =$$

The negative solution is extraneous since we are dealing with length and a negative length would not make sense. Therefore the ratio is equal to the irrational number 1.618… and is most commonly denoted as φ. [22]

There are many interesting findings about the irrational number phi (φ). One of which is the Golden Ratio conjugate. [23] If we find the reciprocal of φ, we end up with 0.618… (Φ).

$$\Phi = = 0.618\ldots$$

$$\Phi = \varphi - 1 = 0.618\ldots$$

Since both these equations are equivalent, we can set

$$= \varphi - 1$$

The inverse of this must be = Φ + 1 which equals 1.618… (φ)

If we raise φ to consecutive powers we see a familiar pattern start to emerge. [24]

$\varphi = 1.618\ldots$

$\varphi^2 = 2.618\ldots$ or $1 + \varphi$

$\varphi^3 = \varphi\cdot\varphi^2 = \varphi(1 + \varphi) = \varphi + \varphi^2 = \varphi + 1 + \varphi$ $= 1 + 2\varphi$

$\varphi^4 = \varphi\cdot\varphi^3 = \varphi(1 + 2\varphi) = \varphi + 2\varphi^2 = \varphi + 2(1 + \varphi)$

$= \varphi + 2 + 2\varphi$ $= 2 + 3\varphi$

$\varphi^5 = \varphi\cdot\varphi^4 = \varphi(2 + 3\varphi) = 2\varphi + 3\varphi^2 = 2\varphi + 3(1 + \varphi)$

$= 2\varphi + 3 + 3\varphi$ $= 3 + 5\varphi$

$\varphi^6 = \varphi\cdot\varphi^5 = \varphi(3 + 5\varphi) = 3\varphi + 5\varphi^2 = 3\varphi + 5(1 + \varphi)$

$= 3\varphi + 5 + 5\varphi$ $= 5 + 8\varphi$

By now we should see that the solution is following the Fibonacci numbers. The following formula represents this:

$$\varphi^n = F_{n-1} + F_n\varphi$$

In the Fibonacci sequence (1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233…) the ratios of the larger to the smaller of consecutive numbers in the series approach the Golden Ratio. [25]

1/1 = 1 2/1 = 2 3/2 = 1.5

5/3 = 1. 8/5 = 1.6 13/8 = 1.625

21/13 = 34/21 = 1. 55/34 = 1.6176… 89/55 =1.6 144/89 = 1.61977… 233/144 = 1.6180

Johannes Kepler proved that the golden ratio is the limit of these ratios found in the Fibonacci sequence. He found that alternately each successive ratio would be higher or lower that the Golden Ratio, but as *n* increase, the ratios approach φ. [26]

= φ

Constructions

Constructing the Golden Cut [27]

Step 1: Begin with a line *l* and a point *P* above the line. With *P* as the center, draw a circle large enough to intersect the line at two points *A* and *F*. Find the midpoint by drawing two intersecting circles of equal radii centered on points *A* and *F*. Connect the points of intersection of the circles with a line. We will call the midpoint *B*.

Figure 4: Finding midpoint of , from "Constructions for the Golden Ratio," www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/phi2DGeomTrig.html#hof2003 (accessed November 29, 2010).

Step 2: Find the midpoint of segment by drawing two intersecting circles of equal radii centered on points *A* and *B*. Connect the points of intersection with a line. This point will be the midpoint of *AB*. We will call it *M*.

Figure 5: Midpoint M, from "Constructions for the Golden Ratio," www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/phi2DGeomTrig.html#hof2003 (accessed November 29, 2010).

Step 3: Draw a circle with radius centered at *B*. The intersection of this circle with the perpendicular at *B* we will name point *D*. = .

Figure 6: Finding D, from "Constructions for the Golden Ratio," www.maths.surrey.ac.uk/hosted-

sites/R.Knott/Fibonacci/phi2DGeomTrig.html#hof2003
(accessed November 29, 2010).

Step 4: Draw line segment. Draw a circle with radius , centered at *D*. The intersection of this circle with we will call point *E*. Draw another circle with radius , centered at *A*. The intersection of this circle with is the Golden Cut *C*.

Figure 7: The final step, from "Constructions for the Golden Ratio," www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/phi2DGeomTrig.html#hof2003 (accessed November 29, 2010).

Constructing the Golden Rectangle [28]

Step 1: Begin with a square *ABCD*. Find the midpoint of (see Step 2 of constructing a Golden Cut). Label the midpoint *M*.
Step 2: Draw a circle with radius , centered at *M*. Extend the line segment through *B* until it intersects the circle. We will call this point *E*. (*B* is not the Golden Cut). Draw line perpendicular to . Extend through *C*. The intersection of the perpendicular and this line will be *F*. The rectangle *AEFD* is a Golden Rectangle. (*EFCB* is also a Golden Rectangle).

Figures 8 – 11: Steps to constructing a Golden Rectangle, from "nrich.maths.org :: Mathematics Enrichment :: December 2010 Front Page," http://nrich.maths.org (accessed November 5, 2010).
Constructing the Golden Spiral [29]

Step 1: Begin with Golden Rectangle. Construct a square by drawing a circle whose radius is the height of the rectangle.
Step 2: Repeat this process inside the smaller Golden Rectangles.
Step 3: Inscribe quarter circles in each square to create the spiral.

Figures 12-15: Steps to constructing a Golden Spiral, from Jill Britton, "Some More Geometry." http://britton.disted.camosun.bc.ca/goldengeom/goldengeom2.html (accessed November 29, 2010).

The Golden Triangle is an isosceles triangle with base angles 72°. When the length of the shorter leg of the triangle bisects one of the base angles, it intersects the opposite leg at a Golden Cut, thus forming a similar acute Golden Triangle. [30] The obtuse triangle that is formed is called the Golden Gnomon. [31]

Figure 16: The Golden Triangle, from "Constructions for the Golden Ratio," www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/phi2DGeomTrig.html#hof2003 (accessed November 29, 2010).

The Pentagon and Pentagram also contain Golden relationships. The ratio of the side of a regular pentagon to its diagonal is φ. When the pentagram is inscribed within the pentagon, many of the ratios between segments are also φ. [32] If a pentagon is divided by diagonals from one vertex, the resulting triangles are Golden. The middle triangle is an acute Golden Triangle and the other two are obtuse Golden Triangles. [33]

Figure 17: The Pentagon and inscribed Pentagram, from R. Knott, D.A. Quinney, and PASS Maths, "The life and numbers of Fibonacci | plus.maths.org," http://plus.maths.org/issue3/fibonacci (accessed November 5, 2010).

Architecture

The Parthenon in Athens is the classic example of the Golden Ratio being used in architecture. It was constructed between 448 – 432 BC as a temple for the goddess Athena. The structure contains a multitude of Golden Rectangles. [34] The Greek sculptor, Phidias, created many pieces for display in the temple. These sculptures also employed the Golden Proportion in their design. [35] The ancient Greeks were very conscious of the aesthetic beauty of the Golden Ratio and used itin many of their architectural and artistic designs.

Figure 18: The Parthenon, from Narain, D.L. "Mr. Narain's Golden Ratio WebSite." http://cuip.uchicago.edu/~dlnarain/golden/ (accessed October 4, 2010).

The Great Pyramid of Khufu in Giza, Egypt is another example of an ancient structure where there seems to have been some knowledge and use of Golden Ratios in its design. It is one of the Seven Wonders of the Ancient World and it is easy to see why. [36] The Pyramid's construction is a geometric wonder. Many of the proportions present adhere to the Golden Ratio and to Pi. The ratio of the slant height (height of the triangular face) to half of the base approximates φ. [37]

Figure 19: The Great Pyramid of Giza, from "Great Pyramid of Giza - Wikipedia, the free encyclopedia." http://en.wikipedia.org/wiki/Pyramid_of_Khufu (accessed October 4, 2010).
Figure 20: Showing proportions, from Samuel Obara, "Golden Ratio in Art and Architecture," http://jwilson.coe.uga.edu/EMT668/EMAT6680.2000/Obara/Emat6690/Golden%20Ratio/golden.html (accessed October 4, 2010).

Many Medieval churches and cathedrals also include evidence of the use of Golden Ratios in their design. In the same way that the Parthenon's facade may be divided by Golden Rectangles, Notre Dame [38] and the Chartres Cathedral in Paris may also be divided. The architects believed in a connection between geometrical design and artistic beauty when incorporating the Golden Ratio into their construction. [39]

Figure 21: Chartres Cathedral, from Samuel Obara, "Golden Ratio in Art and Architecture," http://jwilson.coe.uga.edu/EMT668/EMAT6680.2000/Obara/Emat6690/Golden%20Ratio/golden.html (accessed October 4, 2010).

The Swiss architect, Le Corbusier, created the "modulor" (from the words "modul" meaning ratio, and "or" meaning gold) system for architectural design. [40] He consciously employed the Golden Ratio into his design. He was inspired by the work done by Vitruvius, Leonardo Da Vinci and others who found that the proportions of the human body added beauty and function to architecture. Le Corbusier's Villa Stein in Garches, constructed in 1917 is probably the best example of the Modulor system. Le Corbusier believed that the Golden Ratio and the Fibonacci sequence are "rhythms apparent to the eye and clear in their relations with one another. And these rhythms are at the very root of human activities. They resound in Man by an organic inevitability, the same fine inevitability which causes the tracing out of the Golden Section by children, old men, savages, and the learned." [41]

In modern times, buildings such as the United Nations Headquarters in New York City and the CN Tower in Toronto have Golden Ratios in their design. The United Nations building is three Golden Rectangles on top of one another. The observation deck of the CN Tower is at the Golden Cut of the height of the Tower, making the ratio of its total height to the height above the observation deck φ. [42]

Figure 22: CN Tower, from "Phi: 1.618." http://goldennumber.net/ (accessed October 4, 2010).
Figure 23: United Nations Headquarters, from "Phi: 1.618." http://goldennumber.net/ (accessed October 4, 2010).

Art

The Golden Ratio is present in many portraits. It is not known if this were purposely done as part of the design, or because of the inherent presence of the ratio in the human form. For some it seems that the use of the Divine Proportion was a conscious decision. Leonardo Da Vinci was a prolific artist who employed the Golden Ratio in his sketches, paintings and sculptures. The *Mona Lisa*, the famous portrait of a woman with a coy smile, is embedded with Golden Rectangles. *An Old Man*, a sketch, has many rectangles sketched on it, many of which approximate Golden Rectangles. [43] *The Last Supper*, *Madonna with Child and Saints*, *St. Jerome*, and *The Annunciation* are all works which use the Divine Proportion in their composition. [44] The *Vitruvian Man* is perhaps one of Da Vinci's most well-known drawings. It is his attempt to show the perfection of the human body in all of its Divine Proportions. [45]

Figure 24: *An Old Man*, from Samuel Obara, "Golden Ratio in Art and Architecture," http://jwilson.coe.uga.edu/EMT668/EMAT6680.2000/Obara/Emat6690/Golden%20Ratio/golden.html (accessed October 4, 2010).

Figure 25: *Mona Lisa*, from "Leonardo da Vinci - Wikipedia, the free encyclopedia." http://en.wikipedia.org/wiki/Leonardo_da_Vinci (accessed November 7, 2010).

Figure 26: *The Annunciation*, from "Leonardo da Vinci - Wikipedia, the free encyclopedia." http://en.wikipedia.org/wiki/Leonardo_da_Vinci (accessed November 7, 2010).

The Golden Ratio may also be observed in the works of Botticelli, Michelangelo, Raphael, Salvador Dali, Maurice Seurat, and Piet Mondrian. Golden Triangles and Golden Stars (pentagrams) are used in the composition of several famous paintings including *The Holy Family,* by Michelangelo and *The Crucifixion* by Raphael. Salvador Dali uses the Golden Rectangle as the frame for *The Sacrament of the Last Supper*. He also uses a dodecahedronin his composition. The dodecahedron is made up of twelve pentagons and is closely linked to the Golden Ratio. In *Bathers,* Maurice Seurat, uses Golden Rectangles to divide the focus of his painting. Piet Mondrian, a Dutch painter, created many works involving horizontal and vertical lines which often formed Golden Rectangles. [46]

Figure 27: *The Crucifixion* (Golden Triangle and Star highlighted), from Samuel Obara, "Golden Ratio in Art and Architecture," http://jwilson.coe.uga.edu/EMT668/EMAT6680.2000/Obara/Emat6690/Golden%20Ratio/golden.html (accessed October 4, 2010).

Figure 28: *The Sacrament of the Last Supper*, from Samuel Obara, "Golden Ratio in Art and Architecture," http://jwilson.coe.uga.edu/EMT668/EMAT6680.2000/Obara/Emat6690/Golden%20Ratio/golden.html (accessed October 4, 2010).

Unit

Lesson One

I will begin the unit by displaying six rectangles. Each rectangle will be different, some will be squarer, some skinny, and one will be in the Golden Proportion.Students will create a survey/poll for the most "beautiful" rectangle. Encourage students to include thought-provoking questions such as “Why did you choose that particular rectangle?” and “Do you think a rectangle be beautiful?” in their survey. In pairs, students can be sent to poll other classes. Once the data has been collected, the class will create a bar graph to display the data obtained from the poll. Ask questions about the results of the data such as “Are you surprised by the results?” and “Do you think that the sample size was large enough?” Have students qualify their answers. Discuss the answers to the survey questions. Talk about whether or not the class believes that the survey questions were appropriate. Should the questions be modified if we were to redo the survey with a different group?

The students will take measurements of each of the rectangles and find the ratio of the length to the width of each. Next we will find objects around the classroom that are rectangular. (For example, textbooks, notebooks, TV screen, index cards, desktops, the door). In pairs students will measure using rulers or tape measure the length and width of items. They will then create ratios comparing the longer side to the shorter side. How many of these measurements are close to the students’ choice of the most beautiful rectangle? [47]

Explain that since ancient times (5th Century B.C.) the proportion of 1:1.618… has been considered one of beauty. Euclid was the first to write about this proportion. He said that any line may be cut in such a way that the ratio of the line to its longest part is proportional to the ratio of the longer part to the shorter part. The ratio goes by many names: the Golden Mean, the Golden Section, the Golden Ratio, and the Divine Proportion. [For Algebra extension, see *] This ratio turned out to be an irrational number 1.618… It has since been denoted as the Greek letter “phi,” φ.

*Algebra Extension: At this point, demonstrate how to make the Golden Cut (see Constructions). Label the whole *x*, and the longer segment 1. Have the students come up with an expression for the shorter segment. Using proportions and quadratic formula (see Mathematically Speaking), lead students to discover the value of *x* (i.e. 1.618… = φ). You may wish to show a video clips about how φ is calculated. See “HowStuffWorks Videos "Assignment Discovery: Golden Ratio" [48] and "HowStuffWorks Videos "Assignment Discovery: The Golden Ratio.” [49]

Many throughout history have marveled at the places that this ratio appears in nature. They have also attributed to it beauty. It is a ratio used by the ancient Greeks in their construction of the Parthenon. It was used by the Egyptians in the building of the great pyramids in Giza. There is even evidence that the Mayans used it in their calendar. [50] It has also been used in paintings and sculptures. Even today, we find the Golden Ratio being used by marketers in design.

Give the students an assignment to find six rectangular objects around their house. Take measurements and create ratios of longer side to shorter side. Which objects approximate φ? Have students explain why they think that these objects are made in the Golden Ratio. Do you think that it was done on purpose? What might the purpose be for designing objects in the Golden Proportion?

Lesson 2

Demonstrate how to construct a Golden Cut, a Golden Rectangle, and a Golden Spiral (see Constructions). Have students use a straight edge and a compass for their constructions. No measurements will be taken.

Give students pictures of the Parthenon. Have students draw the length of the base of the Parthenon on a transparency overlaid on the Parthenon. From there students should construct a Golden Rectangle. Students should “spiral” Golden Rectangles. Once students have a variety of sizes they should use their Golden Rectangles to see if the Parthenon is approximately a Golden Rectangle, and how many other Golden Rectangles they can find within its façade.

For an assignment, offer students other pictures of buildings such as Notre Dame, the Chartres Cathedral, the United Nations Headquarters, and the CN Tower. Students should create Golden Rectangles to fit a base length on the picture, like they did with the Parthenon. Do these buildings include Golden Rectangles in their design? How so?

Lesson Three

For this lesson, students should be in groups of three or four. Students will be exposed to a variety of artwork and photography that incorporate the Golden Ratio. Students should discuss what the appeal of each piece is. Is it beautiful? What makes it beautiful? Do you think that the way that it has beenlaid out or composed plays a part in how you perceive it as being attractive? Take up the pictures and replace them with the same pictures that have been cropped differently. [51] How does this affect how you feel about the artwork/photograph? Do you think that the artist/photographer created their composition purposefully?

Have students explore the compositions of the original pictures.Is there evidence of Golden rectangles in the composition of each? [52] [53] Do you think that the artist/photographer was aware of this Golden Ratio when they composed their artwork/photograph? Or was the presence of the Golden Ratio just by chance?

For an assignment, have students take a photograph that demonstrates a conscious effort to incorporate the Golden Ratio. Each student should have the opportunity to present his/her photograph to the class. The student should elaborate about how the Golden Ratio was used. Other students should be allowed to make comments about the use of the Golden Ratio. Suggestions for how the photograph could be improved may be offered. Students may choose to retake their photo. Students work can be displayed.

Lesson Four

Conduct a Socratic Seminar on beauty and how we perceive people to be beautiful. What makes people beautiful? How do we determine who is beautiful? Does the media (magazines and/or television) influence our perception of beauty? Does personality play a part in whether or not we believe someone to be beautiful? Does knowing a person change your perspective on their beauty?

Figure 29: *Vitruvian Man*, from M. Bourne, "The Math Behind the Beauty," http://www.intmath.com/Numbers/mathOfBeauty.php (accessed October 4, 2010).

People throughout history have grappled with similar questions. What makes people beautiful? Not surprisingly, the ancient Greeks and even the Renaissance painters believed that beauty was linked to the Golden Ratio. Men like Phidias (for whom phi is named) believed that by using the Golden Ratio in their sculptures they were approaching the “perfect” form. Leonardo DaVinci emphasized the proportion of the human body in his *Vitruvian Man.* Give pairs of students a picture of the *Vitruvian Man* and have them discover through measurement some of the Golden Proportions present.

The Golden Ratio when found in the proportions of the body is supposed to indicate perfection. This is also true of the proportions of the face. Show the video “The golden ratio in the human face". [54] Investigate this further online at "The Math Behind the Beauty." [55] Many believe that faces that have proportions that approximate φ are beautiful. What do you think, based on our previous discussion? Can math make you beautiful? Or is beauty more than just Divine Proportions?

For an assignment, have students write an opinion paper about beauty. "Is beauty in the eye of the beholder?" They should include information gleaned from class discussion.

Lesson Five

Write down the first five terms in the Fibonacci sequence. Have the students find the next five terms in the sequence. Students should be able to justify/explain how they figured out what came next.

Fibonacci Numbers are quite prevalent in nature. Explain where in nature they are found (pinecones, petals on flowers, sunflower heads, etc.). Show the video "Nature by Numbers." [56]

The Fibonacci Numbers have a special link to the Golden Ratio. Have students find the ratio of a Fibonacci number and the Fibonacci number that came before it for the first 12 terms. The students should notice a pattern. What number is this approaching?

*Algebra Extension: Have the students create a scatter plot of ratios. [57] List 1 should be the first twelve numbers in the Fibonacci sequence {1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144}. List 2 should be the first thirteen numbers without the initial 1: {1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, and 233}. List 3 = List 2/List 1. Students should notice the ratios are close to φ. For the scatter plot, let x = List 2 and y = List 3. Have students estimate a line of best fit before having them calculate it on the graphing calculator. Students should be able to predict what might happen if we added more data to our lists.

Culminating Art Project

Students will choose one of the following art projects to complete the unit:

- Create a Golden Spiral. The Spiral should fill as much of the paper as possible. The Spiral should be colored by square. The inside of the spiral should be a contrasting (opposite) color from the outside of the spiral.

- Create a Fibonacci Spiral. The Spiral should fill as much of a piece of paper as possible. The Spiral should be colored in tints and shades (darkest at the center).

Figure 30: Example of Fibonacci Spiral, from "Photography Lesson: Golden ratio | mean," http://www.photographyicon.com/goldenratio/index.html (accessed October 4, 2010).

- Create a piece of art like Piet Mondrian which incorporates Golden Rectangles and uses only black, white, red, blue, and yellow.

Figure 31: *Composition in Red, Yellow, and Blue*, from Samuel Obara, "Golden Ratio in Art and Architecture," http://jwilson.coe.uga.edu/EMT668/EMAT6680.2000/Obara/Emat6690/Golden%20Ratio/golden.html (accessed October 4, 2010).

- Create a "tree" whose branches and leaves represent the Fibonacci sequence (up to the 7^{th} or 8^{th} term).Each number in the sequence should be represented by a different color.

- Create a portfolio of personal photographs (8 – 12 photos) that incorporates the use of the Golden Triangle, Spiral, or Rectangle.

- Students will be allowed to create a project of their own. It must be approved by the teacher. The project must incorporate the Golden Ratio or Fibonacci Numbers in some way and must be an original work.

Notes

[1] "Phi: 1.618," http://goldennumber.net/ "Quotes related to Phi," (accessed October 4, 2010).
[2] Jason Earls, "Great Mathematical Quotes - Associated Content - associatedcontent.com," http://www.associatedcontent.com/article/701409/great_mathematical_quotes.html?cat=4 (accessed November 7, 2010).
[3] "Bailey Middle School Test Scores - Cornelius, North Carolina – NC," http://www.greatschools.org/modperl/achievement/nc/3383#from (accessed November 7, 2010).
[4] CMS School Improvement Plan 2010-2012: Bailey Middle School 2010-2011.
[5] "Phi: 1.618," "History."
[6] D.E. Joyce, "Euclid's Elements," http://aleph0.clarku.edu/~djoyce/java/elements/elements.html Euclid, *Elements*, Book 6, Definition 3, Proposition 30; Book 2, Proposition 11; Book 4, Propositions 10–11; Book 13, Propositions 1–6, 8–11, 16–18, (accessed November 7, 2010).
[7] Mario Livio, "The golden ratio and aesthetics | plus.maths.org," http://plus.maths.org/issue22/features/golden/ (accessed November 7, 2010).
[8] "Golden ratio - Wikipedia, the free encyclopedia," http://en.wikipedia.org/wiki/Golden_ratio (accessed November 7, 2010).
[9] Toni Beardon, "nrich.maths.org :: Mathematics Enrichment :: Golden Mathematics." http://nrich.maths.org/public/viewer.php?obj_id=2787&part=index&nomenu=1 (accessed November 5, 2010).
[10] Keith Devlin, "Good stories, pity they're not true," http://www.maa.org/devlin/devlin_06_04.html (accessed November 7, 2010).
[11] Beardon.
[12] "Phi: 1.618," "History."
[13] "Phi: 1.618," "Quotes related to Phi."

[14] Samuel Obara, "Golden Ratio in Art and Architecture," http://jwilson.coe.uga.edu/EMT668/EMAT6680.2000/Obara/Emat6690/Golden%20Ratio/golden.html (accessed October 4, 2010).
[15] "Golden ratio - Wikipedia, the free encyclopedia."
[16] J.J. O'Connor and E.F. Robertson, "Golden ratio," http://www-history.mcs.st-andrews.ac.uk/HistTopics/Golden_ratio.html (accessed November 7, 2010).
[17] "Golden ratio - Wikipedia, the free encyclopedia."
[18] "Phi: 1.618," "Quotes related to Phi."
[19] "Phi: 1.618," "History."
[20] Mario Livio, *The golden ratio: the story of phi, the world's most astonishing number*, 5.
[21] "Penrose tiling - Wikipedia, the free encyclopedia," http://en.wikipedia.org/wiki/Penrose_tiling (accessed November 8, 2010).
[22] Sasho Kalajdzievski and R. Padmanabhan, *Math and art: an introduction to visual mathematics*, 14.
[23] "Golden ratio - Wikipedia, the free encyclopedia."
[24] "Golden ratio - Wikipedia, the free encyclopedia."
[25] Beardon.
[26] "Golden ratio - Wikipedia, the free encyclopedia."
[27] Sasho Kalajdzievski and R. Padmanabhan, 15.
[28] Sasho Kalajdzievski and R. Padmanabhan, 17.
[29] Sasho Kalajdzievski and R. Padmanabhan, 18.
[30] Obara.
[31] "Golden ratio - Wikipedia, the free encyclopedia."
[32] R. Knott, D.A. Quinney, and PASS Maths, "The life and numbers of Fibonacci | plus.maths.org," http://plus.maths.org/issue3/fibonacci (accessed November 5, 2010).
[33] Sasho Kalajdzievski and R. Padmanabhan, 21.
[34] "Parthenon - Wikipedia, the free encyclopedia," http://en.wikipedia.org/wiki/Parthenon (accessed October 4, 2010).
[35] "Phi: 1.618," "History."
[36] "Great Pyramid of Giza - Wikipedia, the free encyclopedia," http://en.wikipedia.org/wiki/Pyramid_of_Khufu (accessed October 4, 2010).
[37] Mario Livio, *The golden ratio: the story of phi, the world's most astonishing number*, appendix 3.
[38] "Phi: 1.618," "Architecture."
[39] Obara.
[40] Obara.

[41] "Golden ratio - Wikipedia, the free encyclopedia." [LeCorbusier, *The Modulor*, 25, as cited in Padovan's *Proportion: Science, Philosophy, Architecture*, 316]
[42] Obara.
[43] Obara.
[44] D.L. Narain, "Mr. Narain's Golden Ratio WebSite," http://cuip.uchicago.edu/~dlnarain/golden/ (accessed October 4, 2010).
[45] Obara.
[46] Obara.
[47] Narain.
[48] "HowStuffWorks Videos "Assignment Discovery: Golden Ratio"," Howstuffworks "Video Channel", http://videos.howstuffworks.com/discovery/30610-assignment-discovery-golden-ratio-video.htm (accessed November 7, 2010).
[49] "HowStuffWorks Videos "Assignment Discovery: The Golden Ratio"," Howstuffworks "Video Channel", http://videos.howstuffworks.com/discovery/30014-assignment-discovery-the-golden-ratio-video.htm (accessed November 7, 2010).
[50] Narain.
[51] Laurie Nyveen, "News Photos and Layout: Photography, the Golden Mean, and Geeky Coolness," http://photoinf.com/Golden_Mean/Laurie_Nyveen/News_Photos_and_Layout.htm (accessed October 4, 2010).
[52] Eugene Ilchenko, "Golden Section and Rule of Thirds (Golden Mean, Golden Ratio, Golden Spiral, Golden Proportion, Golden Triangles)," http://photoinf.com/Golden_Mean/Eugene_Ilchenko/GoldenSection.html (accessed October 4, 2010).
[53] "Photography Lesson: Golden ratio | mean," http://www.photographyicon.com/goldenratio/index.html (accessed October 4, 2010).
[54] Yaratilistr, "YouTube - The golden ratio in the human face," http://www.youtube.com/watch?v=eKGTn5pzC_I (accessed November 7, 2010).
[55] M. Bourne, "The Math Behind the Beauty," http://www.intmath.com/Numbers/mathOfBeauty.php (accessed October 4, 2010).
[56] Cristobal Vila, "YouTube - Nature by Numbers," http://www.youtube.com/watch?v=kkGeOWYOFoA (accessed November 7, 2010).
[57] R. Knott, D.A. Quinney, and PASS Maths.

Bibliography

"Bailey Middle School Test Scores - Cornelius, North Carolina - NC."GreatSchools - Public and Private School Ratings, Reviews and Parent Community. http://www.greatschools.org/modperl/achievement/nc/3383#from. HeaderLink (accessed November 7, 2010). Information for my particular school.
Beardon, Toni. "nrich.maths.org :: Mathematics Enrichment :: Golden Mathematics." nrich.maths.org :: Mathematics Enrichment :: November 2010 Front Page. http://nrich.maths.org/public/viewer.php?obj_id=2787&part=index&nomenu=1 (accessed November 5, 2010). Great interactive website dedicated to the Golden Ratio.
Bourne, M. "The Math Behind the Beauty." Interactive Mathematics - Learn math while you play with it! http://www.intmath.com/Numbers/mathOfBeauty.php (accessed October 4, 2010). Interactive site where you can fit a Golden Ratio mask on celebrity faces.
Britton, Jill. "Some More Geometry." Jill Britton's Website. http://britton.disted.camosun.bc.ca/goldengeom/goldengeom2.html (accessed November 29, 2010). This is a good website for constructions.
"Constructions for the Golden Ratio." Two-dimensional Geometry and the Golden section. www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/phi2DGeomTrig.html#hof2003 (accessed November 29, 2010). This is a good website for constructions.
Devlin, Keith. "Good stories, pity they're not true." Mathematical Association of America. http://www.maa.org/devlin/devlin_06_04.html (accessed November 7, 2010). This article refutes many of the claims about the prolificacy of the Golden Ratio.
Earls, Jason. "Great Mathematical Quotes - Associated Content - associatedcontent.com." Associated Content - associatedcontent.com. http://www.associatedcontent.com/article/701409/great_mathematical_quotes.html?cat=4 (accessed November 7, 2010). Great website for quotes by famous mathematicians.
"Golden ratio - Wikipedia, the free encyclopedia." Wikipedia, the free encyclopedia. http://en.wikipedia.org/wiki/Golden_ratio (accessed November 7, 2010). There is an abundance of information on this site.
"Great Pyramid of Giza - Wikipedia, the free encyclopedia." Wikipedia, the free encyclopedia. http://en.wikipedia.org/wiki/Pyramid_of_Khufu (accessed October 4, 2010). I used this site to verify information about the pyramids at Giza.
Ilchenko, Eugene. "Golden Section and Rule of Thirds (Golden Mean, Golden Ratio, Golden Spiral, Golden Proportion, Golden Triangles)." Photography Composition

Articles Library.
http://photoinf.com/Golden_Mean/Eugene_Ilchenko/GoldenSection.html (accessed October 4, 2010). This site has several photos highlighted by Golden Spirals, Triangles, and the "Rule of Thirds."

Joyce, D.E. "Euclid's Elements." Euclid's Elements: Table of Contents. http://aleph0.clarku.edu/~djoyce/java/elements/elements.html (accessed November 7, 2010). This is an online site for Euclid's *Elements*.

Kalajdzievski, Sasho, and R. Padmanabhan. *Math and art: an introduction to visual mathematics*. Boca Raton: CRC Press, 2008. This is a great text for an overview of math in art.

Knott R., D.A.Quinney, and PASS Maths. "The life and numbers of Fibonacci | plus.maths.org." plus.maths.org. http://plus.maths.org/issue3/fibonacci (accessed November 5, 2010). Interesting information about Fibonacci numbers and their relationship to the Golden Ratio.

"Leonardo da Vinci - Wikipedia, the free encyclopedia." Wikipedia, the free encyclopedia. http://en.wikipedia.org/wiki/Leonardo_da_Vinci (accessed November 7, 2010). This was a great site for information about DaVinci and his artwork.

Livio, Mario. "The golden ratio and aesthetics | plus.maths.org." plus.maths.org. http://plus.maths.org/issue22/features/golden/ (accessed November 7, 2010). Mario Livio gives his take on the Golden Ratio and how it relates to aesthetics.

Livio, Mario. *The golden ratio: the story of phi, the world's most astonishing number*. New York: Broadway Books, 2002. This a comprehensive book about the Golden Ratio.

Narain, D.L. "Mr. Narain's Golden Ratio WebSite." :: CUIP - Gateway :: http://cuip.uchicago.edu/~dlnarain/golden/ (accessed October 4, 2010). This site has eight activities related to the Golden Ratio.

"nrich.maths.org :: Mathematics Enrichment :: December 2010 Front Page." nrich.maths.org :: Mathematics Enrichment :: December 2010 Front Page. http://nrich.maths.org (accessed November 5, 2010). This site provides many articles concerning math topics.

Nyveen, Laurie. "News Photos and Layout: Photography, the Golden Mean, and Geeky Coolness." Photography Composition Articles Library. http://photoinf.com/Golden_Mean/Laurie_Nyveen/News_Photos_and_Layout.htm (accessed October 4, 2010). This site has interesting photos that have been cropped differently.

Obara, Samuel. "Golden Ratio in Art and Architecture." Jim Wilson's Home Page. http://jwilson.coe.uga.edu/EMT668/EMAT6680.2000/Obara/Emat6690/Golden%20R

atio/golden.html (accessed October 4, 2010). This has a multitude of examples where the Golden Ratio is found in art and architecture.
O'Connor, J.J. and E.F. Robertson. "Golden ratio." MacTutor History of Mathematics. http://www-history.mcs.st-andrews.ac.uk/HistTopics/Golden_ratio.html (accessed November 7, 2010). This article outlines the history of the Golden Ratio.
"Parthenon - Wikipedia, the free encyclopedia." Wikipedia, the free encyclopedia. http://en.wikipedia.org/wiki/Parthenon (accessed October 4, 2010). This provided additional information about the Parthenon.
"Penrose tiling - Wikipedia, the free encyclopedia." Wikipedia, the free encyclopedia. http://en.wikipedia.org/wiki/Penrose_tiling (accessed November 8, 2010). This site provided further information about Penrose tilings.
"Phi: 1.618. The best source to the golden section, golden mean, divine proportion, Fibonacci series and phi, 1.618. Explore its application to art, design, life, beauty, mathematics, geometry, stock markets, theology, cosmology and more." http://goldennumber.net/ (accessed October 4, 2010). I found a multitude of information on this website.
"Photography Lesson: Golden ratio | mean." Icon Photography School | Online Photography Course. http://www.photographyicon.com/goldenratio/index.html (accessed October 4, 2010). This site has a great picture of the Parthenon with Golden Rectangles overlaid, and a good photo showing Golden Spiral composition.

Websites for Teachers

"HowStuffWorks Videos "Assignment Discovery: Golden Ratio"." Howstuffworks "Video Channel". http://videos.howstuffworks.com/discovery/30610-assignment-discovery-golden-ratio-video.htm (accessed November 7, 2010).
"HowStuffWorks Videos "Assignment Discovery: The Golden Ratio"." Howstuffworks "Video Channel". http://videos.howstuffworks.com/discovery/30014-assignment-discovery-the-golden-ratio-video.htm (accessed November 7, 2010).
"Museum of Harmony and Golden Section." Internet Archive: Wayback Machine. http://web.archive.org/web/20071112052226/www.goldenmuseum.com/index_engl.html (accessed November 7, 2010).
Vila, Cristobal. "YouTube - Nature by Numbers." YouTube - Broadcast Yourself. http://www.youtube.com/watch?v=kkGeOWYOFoA (accessed November 7, 2010).
Vila, Cristobal. "Nature by numbers.The theory behind this movie." Eterea, the online portfolio of Cristobal Vila.

http://www.etereaestudios.com/docs_html/nbyn_htm/about_index.htm (accessed November 7, 2010).

Wahl, Mark. "Mark Wahl Math Teaching - Resources and Books." Mark Wahl Math Teaching - Resources and Books. http://www.markwahl.com/index.php?id=22 (accessed October 4, 2010).

Weavis, Jane. "YouTube - Phinomenal Phi - the Golden Ratio, a short film by Jane Weavis." YouTube - Broadcast Yourself. http://www.youtube.com/watch?v=SFZ3gHWAa-c&feature=player_embedded# (accessed November 7, 2010).

Yaratilistr. "YouTube - The golden ratio in the human face." YouTube - Broadcast Yourself. http://www.youtube.com/watch?v=eKGTn5pzC_I (accessed November 7, 2010).